Qualitative Superposition

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Abstract

Results are presented generalising superposition to non-linear systems by using qualitative differential equations. These are applied to allow the composition and decomposition of qualitative histories. Histories record the qualitative changes in a system over time, and they can be automatically generated by qualitative simulation. The qualitative superposition of such histories is shown to be identical to the qualitative simulation of interactions within linear systems, and many non-linear systems. The result of adding two histories is a partial envisionment for the system, and the recreation of the interaction history is a path traversal of the envisionment space. The technique is useful when a reasoning system needs to decompose an interaction history to identify the state of each contributing history, and when examples of a systems's behaviours exist as histories but no model is available. Formalising the limits to qualitative superposition also has implications for other forms of inference, such as the resolution of multiple causal inferences through a single parameter.

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1 Introduction

Histories record the qualitative changes in a system over time. They were first proposed by Hayes [9] as an improvement on the situational calculus for describing actions and change. A history was “a connected piece of space-time" which recorded events within a restricted spatial extent. The path a ball takes through the air or the response a physiological system makes to a disturbance can be recorded with histories.

Hayes noted that a history could be thought of as the extension or occurrence of a physical process. This relationship gained fuller expression in Forbus’ Qualitative Process Theory [6], in which physical systems were described in terms of the processes that occurred within them. QP Theory proposed a mechanism for the automatic generation of histories from such process descriptions. Since then, the generation of histories from system descriptions has been more formally identified as qualitative simulation, of which Kuipers’ QSIM [10] is a well known example. Histories in QSIM become solutions to models composed of qualitative differential constraints.

The relationship between histories and the underlying system model that produces them is reasonably well understood. Much less is known about how histories relate to one another, and how a reasoning system could use such relationships to make useful inferences. In Hayes’ Manifesto he noted that histories could relate in various ways. They could be adjacent, both spatially and temporally, or they could be hybrid in the sense that histories might interact. When two aircraft collide in mid air, we can think of each as having its own history, and at the moment of impact, a new composite history forms.

The *principle of superposition* is a well known mathematical property of linear systems [13]. It allows any two system states to be added to produce a new state description which is still consistent with that system. While representing a powerful formal technique, superposition is limited because it in general is inapplicable to non-linear systems. It will be show below that by relaxing system descriptions through the use of qualitative mathematical representations, superposition can in fact be applied to a wide range of non-linear systems. This result allows qualitative descriptions of system behaviours - histories - to be added qualitatively to reproduce behaviours consistent with the interaction of those histories.

Hayes [9] saw histories as a “basic ontological primitive” but work since then has focused largely on the mechanisms by which we can generate histories. Histories have thus become a by-product of inference procedures such as qualitative simulation. Qualitative superposition allows histories to once again be considered an important representational form in their own right.

1.1 Motivation

There are several situations in which the superposition of individual histories to determine interactions is advantageous.

Behavioural Decomposition: Qualitative simulation allows the overall behaviour of a system with interacting events to be determined. The simulation of interactions however loses information. We no longer know the individual state of each individual event within the interaction, just the effects of the union through the compound history. This becomes a problem when we want to know what individual processes are up to during the interaction. Such information is necessary for example in process monitoring applications. If the task is to monitor and treat a patient with multiple diseases, the state of each individual disease may need to be tracked within the overall observed behaviour and its individual response to therapy determined [2]. Such state information is inaccessible through normal qualitative simulation.

Behavioural Composition: If we can determine the behaviour of interactions within a system from the participating individual histories, then we only need to generate the individual histories to reason about interactions. Intuitively, this seems to be a simpler method of "naive" reasoning about interactions in physical systems than reasoning from first principles with a model. When using histories as the knowledge primitive, a reasoner can also use extra information to constrain the outcome of a situation. If, for example a bird collides with an aircraft, we can note that the mass of the aircraft completely dominates that of the bird. Of all the potential paths the two might take after the collision, we need only consider the history in which the aircraft's path is unchanged. A full qualitative simulation of the interaction system is avoided, all the inference occurring with the individual histories.

Reasoning without Models: One important consequence of using histories as a representational primitive is that in some circumstances one can make strong conclusions about systems that are as yet unmodelled. For example, empirical information about typical behaviours of a system may be available, but the system may be insufficiently characterised to have a useful qualitative model. Nevertheless qualitative superposition allows predictions to be made about the interactions of multiple behaviours from empirically identified single behaviours. One consequence of the lack of a qualitative model, as will be shown later, is that such predictions may be defeasible.

2 Reasoning with Histories - An Example

Consider the monitoring problem posed in the previous section. A physical system has a number of events occurring concurrently within it, and we wish to monitor the behaviour of each individual event along with the overall behaviour. The behaviour of fluid flowing within a U-Tube will serve as an example. A qualitative model of a U-Tube system is shown in Figure 1. The model is taken from [10] and is composed of a number of qualitative constraints between observable parameters (see section 3). The U-Tube is modelled as two fluid filled tanks (arm A and B) connected by a pipe. We model the fluid within the system with parameters for fluid levels, pressures and flows. These parameters are related to each other through qualitative mathematical constraints like addition, or statements of functional monotonicity.

Figure 1 near here

Qualitative simulations using QSIM on the U-Tube model following single increments of fluid being added to either arm A or B of the U-Tube are presented diagrammatically in Figure 2. Prior to the fluid increments, the system is quiescent, with equal positive fluid levels in each arm, recorded with the labels a(0) and b(0) respectively. To initialise the simulation for an increment to arm A we make the fluid level in arm A take a value greater than a(0) (labelled a(1) in Figure 2.). The simulation then discovers 3 distinct qualitative states, as a flow of fluid occurs between arm A and B, and a new quiescent state is reached with both fluid levels steady at new values a(2) and b(1), both higher than prior to the fluid increment. The label a(2) for the level in arm A lies somewhere between the level before and after initialisation. The simulation output for a single increment to arm B mirrors that for arm A.

Figure 2 near here

If QSIM is initialised to simulate the effects of the two fluid increments simultaneously, three separate histories are generated with quite different final state descriptions (see Figure 3). The histories correspond to the situations in which the amount of fluid added to arm A is equal to, greater than, and less than that added to arm B. The relative height of the initial fluid levels, and the direction of the net fluid flow between the arms is different in each of the three behaviours. In the first the history, the fluid increments are of equal size, so that no net flow occurs at this level of model resolution. In the middle most history the flow rate and pressure difference indicates that a net shift of fluid occurs from arm b to arm a indicating that the amount added to arm b was the greatest. The reverse scenario is represented in the third history.

Figure 3 near here

From the point of view of a reasoning program that wishes to track both contributing events, these three interaction histories are inadequate. Three problems are apparent:

- The loss of information about each individual event within the system description.
- The ambiguity of the possible outcomes. There is no clear way of informing QSIM which of the outcomes is preferred.
- Initialising a function prior to simulation is ambiguous if it is perturbed by *both* events and the initialisations conflict.

What is required is a way of mapping the states of the individual histories in Figure 2 to an interaction state in the histories in Figure 3. In fact such a mapping is possible. As foreshadowed in the introduction, it corresponds to the qualitative addition or superposition of the state descriptions taken from the individual histories to generate a state in the interaction history. Examining the rightmost interaction history in Figure 3, the state additions that produced it are:

$$a1b1 \rightarrow a2b2 \rightarrow a3b3$$

i.e. adding state a1 from the increment to arm A history to state b1 from the increment to arm B history can produce state a1b1 in the interaction history.

The three difficulties presented previously when simulating the progression of any two interacting histories have now been overcome. The first difficulty was the inability to track the effects of individual histories within the concurrent history. This information is now made explicit. The second problem, that of ambiguity of outcome, can be resolved by the use of information about the relative magnitude of the influence of each history on the interaction system during superposition. The process of state addition can be biased to favour solutions consistent with one history dominating the other. Previously there was no way of expressing such information in the standard QSIM formalism. Finally, the third problem of conflicting function initialisations can also be resolved through the use of relative magnitude information.

2.1 Overview

Given that there is some motivation for reasoning with individual histories as a complementary technique to full qualitative simulation, it remains to be determined under what conditions this might be valid. The QSIM algorithm and representation will be used as the exemplar system for the generation of histories throughout the rest this paper. There remains some controversy among researchers working in the area about whether a process ontology like QP theory, or a device ontology [3], [5] are appropriate representations for qualitative reasoning. QSIM makes no assumption about the origin of its qualitative constraint models, and so offers a low level description language for qualitative systems that could fit into either the process or device ontology [8]. The following sections will examine the superposition process in more detail. First, relevant portions of the QSIM formalism will be presented. Next the mathematical basis for qualitative superposition using the QSIM representation will be presented, including non-linear systems. An algorithm for generating composite histories after qualitative superposition is presented next. The paper concludes with a short discussion on reasoning with unmodelled behaviours, the role of superposition in causal reasoning, and future directions for this work.

3 Qualitative Simulation

Since QSIM is used as the reference simulation system, it will be necessary to briefly restate some definitions from Kuipers' original work [10]. Within the QSIM formalism, a system model consists of a number of parameters linked by constraint relationships. QSIM generates a state description for the system consistent with given initial conditions and operating ranges for these parameters, and generates all the valid state descriptions that might follow. A sequence of such states forms a history for the system being modelled. A summary of the QSIM definitions relevant to the results on superposition now follows.

Parameters: A physical system is characterised by a set of real valued parameters $f:[a,b] \rightarrow \Re^*$ which vary continuously over time, are continuously differentiable and have finitely many critical points in any bounded interval. Such parameters are called *reasonable functions*.

Landmarks: Every parameter is associated with a finite set of landmark values which must include 0, f(a), f(b), and the values of f(t) at each of its critical points. Landmark values are considered to be the only interesting values from which a qualitative state description need be drawn.

Time Points: Distinguished time points are those points where something important happens to a parameter, such as passing a landmark value or reaching a limit. Reasonable functions have a finite set of distinguished time points and landmarks values.

Qualitative State: The qualitative state of a parameter consists of its ordinal relation within the landmark values and its direction of change. If $l_1 < \bullet\bullet\bullet < l_k$ are the landmark values of $f:[a,b] \rightarrow \Re^*$, then for any $t \in [a,b]$, the qualitative state of f at t is $QS(f,t)$ and is a pair $(qval,qdir)$ defined as follows:

$$qval = \begin{cases} l_j & if \quad f(t) = l_j \\ (l_j, l_{j+1}) & if \quad f(t) \in (l_j, l_{j+1}) \end{cases}$$

$$qdir = \begin{cases} inc & if \quad f(t) > 0 \\ std & if \quad f(t) = 0 \\ dec & if \quad f(t) < 0 \end{cases}$$

Qualitative Behaviours: The qualitative behaviour of f on $[a,b]$ is the sequence of qualitative states of f:

$$QS(f,t_0), QS(f,t_0,t_1), QS(f,t_1), \ldots, QS(f,t_{n-1},t_n), QS(f,t_n)$$

alternating between qualitative states at distinguished time points, and qualitative states on intervals between distinguished time points.

Systems: A system is a set of reasonable functions that are related by a set of qualitative differential equations, and the behaviour of the system is the union of the behaviours of its functions. Every qualitative state has a qualitative description, and that description can change at distinguished time points, and remains constant on the open intervals between them. The qualitative state of a system F of m functions is the m-tuple of individual states:

$$QS(F,t_i) = [QS(f_1,t_i), \ldots, QS(f_m,t_i)]$$

The qualitative behaviour of F is the sequence of states:

$$QS(F,t_0),QS(F,t_1),...,QS(F,t_n).$$

Constraints: A system model involves detailing functional relationships through a number of qualitative functional constraints. Constraints may be two or three placed predicates, and they restrict the qualitative values that may be assigned to functions. The QSIM constraints are:

- Add(f,g,h) iff $f(t) + g(t) = h(t)$
- Mult(f,g,h) iff $f(t) \times g(t) = h(t)$
- Minus(f,g) iff $f(t) = - g(t)$
- Deriv(f,g) iff $f'(t) = g(t)$
- $M^+(f,g)$ iff $f(t) = H(g(t))$;$H(x)$ is strictly monotonically increasing
- $M^-(f,g)$ iff $f(t) = H(g(t))$;$H(x)$ is strictly monotonically decreasing

We can consider these constraints as qualitative abstractions of Ordinary Differential Equations (ODEs). We can decompose any ODE into a set of simultaneous first order equations which can be replaced by a qualitative constraint. Thus a system model can be considered to be a set of q*ualitative differential equations* (QDEs). Any behaviour that satisfies an ODE must satisfy the corresponding QDE. Since many possible functions can map onto the same qualitative constraint however, a given QDE may be the abstraction of numerous ODEs.

4 Qualitative Superposition

Just as a qualitative model can be an abstraction of ordinary differential equations, a history can be regarded as an abstraction of the solution to such a set of equations. Two histories can add to reproduce an interaction history precisely when two qualitative solutions for a system can be added to produce another solution to a system[1]. It is a standard result from the study of ordinary differential equations that for a given linear system of differential equations, the sum of any two solutions to the system is also a solution. This superposition property follows from the existence of linearly independent solutions (Appendix A.1). Thus whenever two histories are combined and the histories are from a linear system, a true solution to that system results. However, this is simply not true in general for non-linear systems. Yet it is precisely nonlinear systems that are of interest since most of the physical systems that are of importance display non-linear behaviour.

One way of extending the property of superposability to non-linear systems is to weaken our requirements. In particular if we require superposition to only successfully operate on the qualitative behaviours of functions rather than their strict numeric values, we may still be able to derive useful inferences with it. The

1. In general, when two histories interact, this need not occur over all parameters in each history. This is especially the case when two subsystems communicate with each other through a small subset of their parameters. The interaction occurs only through the shared parameters, and this is specifically the area we are interested in. A simplifying assumption for the remainder of this discussion will be that non-shared parameters can be treated as a 'black box' when dealing with interacting subsystems and their interacting histories. We will only be interested in the inputs from the non-shared parameters into the shared system.

approach taken below will be to identify conditions in which qualitative solutions from non-linear systems behave as if the system they came from was linear.

Superposition has varying degrees of validity, depending on whether the system it is applied to is linear or non-linear, whether quantitative or qualitative solutions are desired, and on the nature of the non-linearity. Four validity classes can be identified:

Type 1. The Principle of Superposition applies uniformly for the quantitative values of all functions in a system. All linear systems fall into this class.

Type 2. Superposition applies for qualitative values of all functions in a system. Functional relations are preserved between the values produced by superposition.

Type 3. Functional relations cannot be maintained following qualitative superposition, but the resulting solution demonstrates the correct qualitative behaviour. This means that the sign of a function and its derivative can be predicted, but not its ordinal position in a landmark list.[1]

Type 4. Qualitative superposition is not supported.

4.1 General Approach

In the general case a physical system may be represented by an nth order differential equation. It was noted above that such an equation can be reduced to a series of first order equations, which can each be mapped onto QSIM constraints (Appendix A.2). The problem now reduces to identifying conditions under which we can add solutions for each of the qualitative constraints, and derive solutions that are still valid for these constraints. We shall call a constraint that satisfies such a condition a r*easonable constraint* and define it thus:

Definition 4.1: Let C be a qualitative constraint of the form $C(a,b)$ where a and b are reasonable functions. If $a_1, a_2,.. a_n$ are solutions for a and $b_1, b_2,.. b_n$ are solutions for b such that $C(a_1,b_1), C(a_2,b_2),.. C(a_n,b_n)$ are true, then if for any two constraint solutions x, y where $x \in [0,n]$ and $y \in [0,n]$, and $C(a_x + a_y, b_x + b_y)$ is true, then C is a *reasonable constraint*. ❑

A similar definition can be formulated for three placed predicates.

Definition: 4.2: If F is a system of reasonable constraints, then any behaviour produced with F is a r*easonable history*. ❑

4.2 Superposition with QSIM constraints

It can trivially be shown that the QSIM constraints MINUS, ADD, DERIV are reasonable since these constraints capture linear behaviour. Solutions will thus be of Type 1 (or 2 if so desired). Non-linear behaviours lie in the monotonic and multiplication constraints. We can demonstrate that the M^+ and M^- constraints support Type 2 qualitative superposition. This is not the case for the MULT constraint. However, making an assumption about the relative magnitude of the two solutions being combined is a sufficient condition for the MULT constraint to

1. In QSIM, corresponding values cannot be supported for Type 3 solutions.

allow Type 3 solutions to be produced. The following proofs demonstrate the degree of superposition possible for each of the MINUS, ADD, DERIV, M^+, M^- and MULT constraints. The constraint definitions given here are taken directly from [10].

4.2.1 The Linear Constraints

Definition 4.3: MINUS(f,g) is a two placed predicate on reasonable functions f,g:$[a,b] \rightarrow \Re^*$ which holds iff $f(t) = -g(t)$ for every $t \in [a,b]$

Proposition 4.3: If MINUS(f_1,g_1) and MINUS(f_2,g_2) are true then MINUS($f_1 + f_2, g1 + g_2$) holds.

Proof: Given $f_1 = - g_1$ and $f_2 = - g_2$, then

$$\begin{aligned} f_1 + f_2 &= -g_1 - g_2 \\ &= -(g_1 + g_2) \end{aligned}$$

❑

Definition 4.4: ADD(f,g,h) is a three placed predicate on reasonable functions f,g,h:$[a,b] \rightarrow \Re^*$ which holds iff $f(t) + g(t) = h(t)$ for every $t \in [a,b]$.

Proposition 4.4: If ADD(f_1,g_1,h_1) and ADD(f_2,g_2,h_2) then ADD($f_1 + f_2, g_1 + g_2,h_1 + h_2$) holds.

Proof: This result clearly follows from the definition of ADD. ❑

Definition 4.5: DERIV(f,g) is a two placed predicate on reasonable functions f,g:$[a,b] \rightarrow \Re^*$ which holds iff $f'(t) = g(t)$ for every $t \in [a,b]$

Proposition 4.5: If DERIV(f_1,g_1) and DERIV(f_2,g_2) are true then DERIV($f_1 + f_2,g_1 + g_2$) holds.

Proof: Given $g_1 = \frac{df_1}{dt}$ and $g_2 = \frac{df_2}{dt}$, then

$$\begin{aligned} \frac{d}{dt}(f_1 + f_2) &= \frac{df_1}{dt} + \frac{df_2}{dt} \\ &= (g_1 + g_2) \end{aligned}$$

❑

4.2.2 The Non-Linear Constraints

Definition 4.6: M^+ is a two placed predicate on reasonable functions $f,g:[a,b] \rightarrow \Re^*$. M^+ is true iff $f(t) = H(g(t))$ for all $t \in [a,b]$, where H is a function with domain $g([a,b])$ and range $f([a,b])$, differentiable and with $H'(x) > 0$ for all x in the interior of the domain.

The essential property captured in a monotonic increasing relationship between two functions is that they reach critical points at identical distinguished time points and that in the intervening regions they share the same direction of change [10] i.e. $\frac{f}{g'} > 0$. It is this functional relationship that needs to be preserved after superposition.

Proposition 4.6: Given $M^+(f,g)$ and two solution pairs (f_1,g_1) and (f_2,g_2) then we can show $M^+(f_1 + f_2, g_1 + g_2)$ also holds. Thus we seek to demonstrate a functional relationship exists between (f_1,g_1) and (f_2,g_2) such that:

1. $\frac{f'_1 + f'_2}{g'_1 + g'_2} > 0$ between critical points (where $g'_1 + g'_2$ is defined) and
2. $f'_1 + f'_2$ and $g'_1 + g'_2$.reach critical points at the same time.

Proof:

1. Behaviour between critical points

Given that $f_1 = H(g_1)$ and $f_2 = H(g_2)$, we note that

$$f_1 + f_2 = H(g_1) + H(g_2)$$

$$f'_1 + f'_2 = H'(g_1) \times g'_1 + H'(g_2) \times g'_2$$

$$\frac{f'_1 + f'_2}{g'_1 + g'_2} = \frac{H'(g_1) \times g'_1}{g'_1 + g'_2} + \frac{H'(g_2) \times g'_2}{g'_1 + g'_2}$$

Let $A = \frac{H'(g_1) \times g'_1}{g'_1 + g'_2}$ and $B = \frac{H'(g_2) \times g'_2}{g'_1 + g'_2}$. We now evaluate the sign of the expression $\frac{f'_1 + f'_2}{g'_1 + g'_2}$ for values of g'_1 and g'_2:

Case 1.Let $g'_1 = 0$. Then $A = 0$ and $B = H'(g_2)$ which we know from Definition 4.6 is always positive.

Case 2.Let $g'_2 = 0$. Similarly, $B = 0$ and $A = H'(g_1)$ which is also always positive.

Case 3.Let $g'_1, g'_2 > 0$. Trivially both A and B are positive.

Case 4.Let $g'_1, g'_2 < 0$. Noting again that $H'(g_x) > 0$ both A and B evaluate to positive expressions.

Case 5.Let $g'_1 > 0, g'_2 < 0$ or $g'_1 < 0, g'_2 > 0$ and let x and y take the values 1 or 2. In both cases A + B is positive when

1. $|g'_x| > |g'_y|$ and
2. $|H'(g_x)| > |H'(g_y)|$. This second condition is equivalent to stipulating that $\left|\frac{f_x}{g'_x}\right| > \left|\frac{f_y}{g'_y}\right|$, or noting that condition 1 must also hold, simply that $|f_x| > |f_y|$.

These two conditions - $|g'_x| > |g'_y|$ and $|f_x| > |f_y|$ - together imply that the effect of one function's values dominates the others during the superposition and collectively form the *relative magnitude constraint*.

Cases 1 to 5 are summarised in Table 1. It may be possible to specify further conditions for superposition in Case 5 by closer examination of the function H. For example if we know it be linear, then the magnitude constraint may be dropped. If it is weakly linear such that $|g'_x| > |g'_y|$, but that $|H'(g_x)|$ is only minimally *less* than $|H'(g_y)|$ then the expression $H'(g_x) \times g'_x$ may still evaluate to be greater than $H'(g_y) \times g'_y$ and satisfy the sign of derivative requirement.

2. Behaviour at critical points.

We need to show that $f'_1 + f'_2$ and $g'_1 + g'_2$ reach zero at the same time. There are two cases at which $g'_1 + g'_2 = 0$ that need to be examined:

1. Both original functions reach critical points at the same time. By definition, when $g'_x = 0$ then $f'_x = 0$. Thus when f'_1 andf'_2 are zero, $g'_1 + g'_2 = 0$ and the expression $\frac{f'_1 + f'_2}{g'_1 + g'_2}$ is defined and equal to 0, implying that both $f'_1 + f'_2$ and $g'_1 + g'_2$ reach critical points when the original functions reach a critical point.

2 When $|g'_1| = |g'_2|$ but are opposite in sign, then $g'_1 + g'_2 = 0$. However $f'_1 + f'_2$ may be non-zero, making $\frac{f'_1 + f'_2}{g'_1 + g'_2}$ undefined. In this case one superposed function may reach a critical point without the other doing so. This case is avoided by the stipulation in Case 5 of the first part of this proof that a relative magnitude constraint must be enforcable for superposition to be possible when functions being added are of opposite sign.

A similar proof can be constructed for the M^- constraint. ❑

	+	✓	✓	*rmc*
g₁′	**0**	✓	✓	✓
	-	*rmc*	✓	✓
		+	**0**	**-**
			g₂′	

Table 1. Summary of Type 2 qualitative superposition results using the monotonic increasing constraint, given M^+(f1 + f2,g1 + g2). (rmc indicates that the relative magnitude constraint applies).

Definition 4.7: MULT(*f,g,h*) is a three placed predicate on reasonable functions *f,g,h*:[*a,b*] $\rightarrow \Re^*$ which holds iff $f(t)\times g(t) = h(t)$ for every $t \in [a,b]$.

In the general case if $f_i \times g_i = h_i$ and $f_j \times g_j = h_j$ then $(f_i + f_j) \times (g_i + g_j) \neq (h_i + h_j)$. However, we are interested in identifying situations in which the expression is still qualitatively true i.e. $h_i + h_j$ displays the same qualitative behaviour as $(f_i + f_j) \times (g_i + g_j)$ to produce a Type 3 prediction. This reduces to two subproblems:

- When is the sign of the function value preserved?
- When is the sign of derivative preserved?

Proposition 4.7: If MULT(f_1,g_1,h_1) and MULT(f_2,g_2,h_2) are true and we can specify the relative magnitude constraint, or a similar equivalence constraint i.e.:

1. $(|f_1(t)| > |f_2(t)|, |f'_1(t)| > |f'_2(t)|$ and $|g_1(t)| > |g_2(t)|, |g'_1(t)| > |g'_2(t)|)$
   or
   $(|f_1(t)| < |f_2(t)|, |f'_1(t)| < |f'_2(t)|$ and $|g_1(t)| < |g_2(t)|, |g'_1(t)| < |g'_2(t)|)$
2. $(|f_1(t)| = |f_2(t)|, |f'_1(t)| = |f'_2(t)|$ and $|g_1(t)| = |g_2(t)|, |g'_1(t)| = |g'_2(t)|)$

for all $t \in [a,b]$ then

- $sign(h_1 + h_2) = sign((f_1 + f_2) \times (g_1 + g_2))$ and

- $sign\left(\frac{d}{dt}(h_1 + h_2)\right) = sign\left(\frac{d}{dt}(f_1 + f_2) \cdot (g_1 + g_2)\right)$

Proof: Consider Case 1 (relative magnitude constraint). Note the result for two real numbers *a, b* that if $|a| > |b|$ then *sign(a + b) = sign(a)*.

$$
\begin{aligned}
& sign\,((f_1 + f_2) \cdot (g_1 + g_2)) \\
&= sign\,(f_1 \cdot g_1) \\
&= sign\,(f_1 g_1 + f_2 g_2) \\
&= sign\,(h_1 + h_2)
\end{aligned}
$$

Further, let

$$
\begin{aligned}
x &= (f_1 + f_2) \cdot (g_1 + g_2) \\
x' &= (f_1 + f_2)' \cdot (g_1 + g_2) + (f_1 + f_2) \cdot (g_1 + g_2)' \\
&= (f_1' + f_2') \cdot (g_1 + g_2) + (f_1 + f_2) \cdot (g_1' + g_2') \\
sign\,(x') &= sign\,(f_1' \cdot g_1 + f_1 \cdot g_1')
\end{aligned}
$$

Let

$$
\begin{aligned}
y &= h_1 + h_2 \\
&= (f_1 g_1 + f_2 g_2) \\
y' &= f_1' g_1 + f_1 g_1' + f_2' g_2 + f_2 g_2' \\
sign\,(y') &= sign\,(f_1' g_1 + f_1 g_1') \\
&= sign\,(x')
\end{aligned}
$$

A similar argument can be produced for the equivalence constraint defined in Proposition 4.7, case 2. ❑

5 Generating Composite Behaviours

Having demonstrated that qualitative superposition is valid for most systems expressible in the QSIM representation, it is now necessary to demonstrate how this property can be harnessed to generate interaction histories. Given two reasonable histories originating from the same system, how do we actually generate a composite behaviour to represent their interaction? There are two fundamental steps to this process:

1. *Superposition*: Adding qualitative states to produce all possible composite states,

2. *Assembly*: Concatenating composite states into a legal behaviour for the system.

The soundness of addition for reasonable histories means that any two such histories derived from the same set of QDEs can be used. An order preserving addition of states needs to be performed i.e. a state from one history is added to all the states in the other history with which it might possibly interact.

However, addition is an underconstrained process. Other solutions are also produced, along with the correct one, because of the inherent ambiguity of qualitative addition. Adding $QS_a(0/\infty,inc)$ to $QS_b(0/\infty,dec)$ produces either $QS_{a+b}(0/\infty,inc)$, $QS_{a+b}(0/\infty,std)$ or $QS_{a+b}(0/\infty,dec)$. Further, such solutions could be assembled into indefinitely long behaviours e.g.

$QS_{a+b}(<0/\infty,inc>,t_n)$, $QS_{a+b}(<0/\infty,std>,t_n/t_{n+1})$, $QS_{a+b}(<0/\infty,dec>,t_{n+1})$, $QS_{a+b}(<0/\infty,std>,t_{n+1}/t_{n+2})$, $QS_{a+b}(<0/\infty,inc>,t_{n+2})$, $QS_{a+b}(<0/\infty,std>,t_{n+2}/t_{n+3})$,...

This problem is called "chatter" [11], and results from the inherent ambiguity of qualitative simulation, as well as problems with the locality of transition value selection in QSIM. These problems can be controlled by:

- Filtering composite states that are not valid system behaviours during the qualitative addition.
- Enforcing behavioural continuity when assembling sum states into new histories.

Both of these techniques will now be explored in more detail.

5.1 Superposition - Creating and Filtering Composite States

Although adding two qualitative states taken from two histories will produce unwanted states along with the real ones, there are at least two ways that such states can be eliminated:

1. *Using the original system constraints*. The sum states can be filtered through the system model that produced the original histories i.e. a composite behaviour must be legal for the system that produces it.
2. *Making an assumption about relative magnitude*s. Knowing the magnitude of the effect each history has on the interaction allows a decision about which history dominates during the addition to be made. While this is required to establish reasonableness for some non-linear systems, it also eliminates spurious solutions during state addition.

Rather than generating all possible states derivable from qualitative addition and then filtering them, it is easier to filter them during the process of qualitative addition. In effect, a single pass qualitative simulation is performed using the system model along with addition constraints between the behaviours of interest.

Further, rather than performing a full permutation of state additions, only those states that preserve functional or temporal continuity need to be added (see Section 5.2). In particular, transients from contributing histories cannot exist more than momentarily. Such transients, when they occur, are present as initial states of the histories being added. We avoid adding initial states with transients to states from the other history other than its initial state. Hence states a1 and b1 in Figure 2. can only be assembled into composite state a1b1. A state a1b2 would imply that a history of a1b1 → a1b2 was possible, implying incorrectly that the a1 transient could extended over the two qualitative states, a point and an interval.

The state generation algorithm first selects the states to be added, and then performs the addition by simulation. Once two states have been selected for addition, the superposition by qualitative simulation occurs in three stages. Assume a system of functions *F*, a set of constraints *C*, and two reasonable histories H_1 and H_2 from *F* and *C*. For each pair of qualitative states to be added we perform the following steps:

Step 1 - Composite Model Assembly:

1. For each function $f \in F$ create three copies labelled f_1, f_2, f_{sum}. Call the system of new functions F_{sum}.
2. For each function triplet so created, generate an *ADD* constraint $ADD(f_1,f_2,f_{sum})$.
3. For each constraint $c \in C$ generate a copy *csum* between the sum functions F_{sum}. Call the new *ADD* constraints and the copy of *C* together C_{sum}.
4. Landmark values, domain restrictions and corresponding values for each $f \in F$ are transferred for each copy function $f_{sum} \in F_{sum}$.

Step 2 - Composite Model Initialisation:

For each function $f_1 \in F_{sum}$ initialise the function value to its value in H_1. Repeat the procedure for each $f_2 \in F_{sum}$ using values in H_2.

Step 3 - Qualitative Simulation on the Composite Model:

Using QSIM, perform a single iteration of the algorithm on the composite model F_{sum} and C_{sum} to find legal values for each *fsum* and assemble these values into legal state descriptions for F_{sum}. Each coherent set of value assignments to the members of F_{sum} is a solution to the addition.

During the qualitative simulation on the composite system model, a restricted table of possible solutions for the *ADD* constraint can be used when one state in the addition is known apriori to dominate the addition (Tables 2 and 3). Such knowledge is needed when superposition requires a relative magnitude constraint, but can also be used to assist in reducing the possible states derived during the simulation.

The result of this process for two states taken from two reasonable histories is all the possible interaction states derivable for those two states. The next step in the

+ *P* *Q*	>0	0	<0
>0	>0	>0	>0
0	-	0	-
<0	<0	<0	<0

Table 2. Landmark addition table with relative magnitude assumption |*Q*| > |*P*|

+ *deriv(P)* *deriv(Q)*	inc	std	dec
inc	inc	inc	inc
std	-	std	-
dec	dec	dec	dec

Table 3. Direction of change addition table with relative magnitude assumption |*deriv(Q)*| >|*deriv(P)*|

superposition process is the linking of these states into complete histories representing interaction behaviours.

5.2 Path Assembly - Enforcing Behavioural Continuity

The technique of e*nvisioning* generates all the possible state descriptions for a system model, based on a set of background assumptions. Any specific history for the system corresponds informally to a path through that envisionment. If we have two reasonable histories, and do a permuted state addition between the two, a state space is defined which is equivalent to or is some subset of a full envisionment. The task of assembling an interaction history from this new state space thus becomes one of path traversal.

A *Logic of Occurrence* was described by Forbus [7], which presented a formalised relationship between histories and envisionments. In particular, the task of selecting a path through the envisionment for a partially created history was described. In our case, if we have selected a state from the partial envisionment created by history addition, we need to determine which states are possible next states for the interaction history. Often additional information about the speed of completion of each history and the synchronicity of transition to subsequent states will be available. Such information greatly constrains the search for a path through the partial envisionment.

A method for path traversal that allows information about the individual interacting behaviours to be utilised will be presented below. In accordance with Forbus' Logic of Occurrence, we assume here that the histories we will generate are finite - that they will terminate at some time in the future. We will also assume Forbus' terminology, which is briefly defined now.

Envisionment: An envisionment ε represents all possible qualitative states a particular system may take on and all legal transitions between them.

Transition functions: These describe all transitions involving a state s. State $s_i \in Befores(s)$ when the envisionment contains a transition from s_i to s. State $s_i \in Afters(s)$ when the envisionment contains a transition from s to s_i.

Paths: A Path is a sequence of states $s_i, .., s_n$ such that $s_{i+1} \in Afters(s_i)$.

Status of States: Consider a history H involving envisionment ε. For every state $s \in$ States(ε), exactly one of the following is true: *possible(s,H), required(s,H), excluded(s,H)*. Simply, a state is *possible* if it may appear in a history, *excluded* if it never occurs, and r*equired* if it must appear.

Forbus introduces mechanisms for handling cycles, and avoiding impossible cyclic behaviours. These techniques for path traversal of envisionments are directly applicable to the state space generated by summing behaviours, and will not be reproduced here.

5.2.1 Path creation

Unlike the envisionments in Forbus' work, states produced by superposition are not explicitly linked. The process of history generation by path traversal must thus include a step that identifies candidate next states. Given a current state *current(s)* in a history H the next states in the path is defined as:

$$\forall\ current(s)\ \ \exists\ next(r) \rightarrow r \in States(\varepsilon), \neg\ excluded(r,H).$$

We need to identify next states that are legal transitions, before we can decide whether they are excluded. Possible states are identified using restrictions imposed by temporal continuity, and excluded states by application of rules that ensure functional continuity through the assembled history.

5.2.2 Temporal Continuity

Next states can be identified with knowledge about the temporal behaviour of the original contributing histories. In particular, each sum state carries with it the labels of the distinguished time points or intervals taken from its contributing states. Next states are identified with reference to that label.

The search for legal next states can be coded in several rules. Assume two reasonable histories H_i and H_j and states drawn from each history s_i and s_j. Let s_i have qualitative description $QS(f_i,t_i)$ and s_j have $QS(f_j,t_j)$, and call the resulting set of summation states $S_{i,j}$. Each state $s \in S_{i,j}$ is then labelled $\mathrm{QS}(\mathrm{f}_{i+j},\mathrm{t}_i,\mathrm{t}_j)$.

Rule 1 - Temporal Progression: For a state $s \in S_{i,j}$ which is the current last state in a history H, and has qualitative state $QS(f_{i+j},t_i,t_j)$, then

$$possible(H,s) = \{QS(f_{i+j},t_i,t_j), QS(f_{i+j},t_{i+1},t_j), QS(f_{i+j},t_i,t_{j+1}), QS(f_{i+j},t_{i+1},t_{j+1})\}$$

The Temporal Progression rule prevents local cycles being generated, and enforces the intuition that for the interaction history to progress to a new qualitative state, then either or both of the contributing histories must also progress. Note that two sequential states in a path may be the result of the same state addition i.e. a state with a $QS(f_{i+j},t_i,t_j)$ label can be a next state for a state with the same label. This reflects the possibility that the addition of two states can produce more than one interaction state.

The Temporal Progression rule can be specialised in some cases to utilise additional knowledge about the relationship between interacting histories. In particular information about synchronicity of history evolution and speed of completion may be utilised to constrain the process of path creation:

Rule 2 - Synchronous Progression: Assume two reasonable histories H_i and H_j have equal path length. For a state $s \in S_{i,j}$ which is the current last state in a history H, and has qualitative state $QS(f_{i+j},t_i,t_j)$, then

$$possible(H,s) = QS(f_{i+j},t_{i+1},t_{j+1})$$

If we know that the interacting histories will both progress to a new distinguished time point in a synchronous manner, then we can exclude asynchronous combinations.

Rule 3 - Asynchronous Progression: For a state $s \in S_{i,j}$ which is the current last state in a history H, and has qualitative state $QS(f_{i+j},t_i,t_j)$, then

$$possible(H,s) = \{QS(f_{i+j},t_{i+1},t_j), QS(f_{i+j},t_i,t_{j+1})\}$$

The Asynchronous Progression Rule enforces asynchronous progression of the contributing histories.

Rule 4 - Speed of Progression: Assume two reasonable histories H_i and H_j of equal path length where for any interval (t_i,t_{i+1}),

$$duration(H_i,t_i,t_{i+1}) < duration(H_j,t_i,t_{i+1}).$$

Then, for any distinguished time point with state $s \in S_{i,j}$ and with qualitative value

$$QS(f_{i+j},t_a,t_b),\ t_a \geq t_b.$$

The speed rule ensures that the faster of the two contributing histories reaches its final state first. In applying the Speed rule, we are making assumptions about which particular path among several alternatives we take to create our composite history. The effect of this rule is not to eliminate possible histories, as the synchronicity rules do, but to eliminate alternative paths for each history. It reduces the size of the search space in the sum envisionment, not the possible histories the envisionment contains.

5.2.3 Functional Continuity

Once the set of possible next states has been identified, the path creation algorithm chooses only those states that maintain functional continuity. For example, two states would not be temporally adjacent in a history if a functional discontinuity exists between them e.g. QS(<0,inc>,t_i), QS(<0,dec>,t_{i+1}) would not be allowed.

Further, continuity determines whether adjacent states can have the same description. In particular, a transition from an interval to a time point should not have the same qualitative state since distinguished time points represent critical points, but the reverse is allowable [10].

Functional continuity applies not just to the history being assembled, but to the histories that contribute to the assembly. The process of selecting possible next states ensures that the contributing histories are traversed in a legal temporal sequence. Along with the composite history, we need to ensure that states from the contributing histories are not prolonged unnecessarily. As indicated in Section 5.1 transients in the histories contributing to the superposition can only exist for an instant and state combinations that would prolong them are filtered during the initial state generation.

Functional continuity could also be invoked in the process of *value recovery*. As we saw in section 4.2.2, there are some values for functions in a monotonic relationship which prevent superposition. However, by looking ahead to the next state, it may be that superposition is once again possible. Where this is the case, the non-superposable state might be recreated by interpolation between its predecessor and successor states, relying on the fact that function values must be continuous.

6 Qualitative Superposition - some consequences

A number of consequences arise from the results in this paper which both expand the repertoire of qualitative reasoning systems and also impose limits on some types of causal reasoning. Two particular issues are explored here - the notion that histories can be used independently of deeper qualitative models, and the validity of the assumption that causal influences can be considered linearly independent.

6.1 Histories without Models

In some circumstances we might want to make predictions based on histories that were not generated from a model. This might be the case when empirical information about a system's behaviour is available but there is insufficient understanding of its mechanisms to construct a qualitative model.

In such circumstances we do not know whether the histories we are manipulating arise from linear or non-linear systems, or in fact if indeed the histories come from the same system. If the behaviours represent faulty behaviours, it may be that a different fault model produced each behaviour. Different models clearly imply different sets of system equations, and in such situations summing solutions is meaningless. A consequence of these uncertainties is that, although a mechanism now exists to produce predictions about interactions directly from behaviours, such predictions must be considered *defeasible*. Recall that although QSIM generates some behaviours that are not true for a system, it will always generate all possible true behaviours. We now have no such guarantee.

For a non-modelled behaviour to be useful, it should be *well-formed*. By this we mean that it displays all the features of a true model generated behaviour such as continuity of function behaviour. Further, it is ideal if a relative magnitude assumption between the summed behaviours can be made. This is so for two reasons. The absence of a model means that the addition of qualitative states is underconstrained - we do not have a model to filter out ill-behaved solutions. The use of relative magnitude information sufficiently constrains the addition process in such circumstances. Secondly, we should make a worst case assumption that the underlying system model is non-linear. By assuming that one history dominates the superposition we ensure that valid predictions are made for systems that contain multiplicative constraints between functions.

6.2 Combining Causal States

Causality may be represented as the temporal progression of states within a system. For example, take a CASNET [14] link between two states involved in the progression of the eye disease glaucoma:

Elevated Pressure Transmitted to Optic Nerve Head $\rightarrow$ (0.9)

Decreased Blood Supply to Optic Nerve Head

Paths through causal networks composed of such states are direct analogues of the qualitative histories discussed in this paper. It may be the case that the effects of two states exist concurrently, and are resolved to produce a third in a network - a process analogous to superposition. If we do view the combination of states as qualitative superposition, then the results presented earlier establish that a number of conditions need to be met prior to combination. In particular the states must be:

1. *Derived from the same structural model*: It is meaningless to use superposition on states that come from different models of a system - the procedure assumes that they represent solutions to the same set of qualitative differential equations. Consequently when states are combined they must belong to the same model. States that come for example, from different fault models of a mechanism cannot be combined.
2. *Satisfy conditions of reasonableness*: If the system that produced the causal states is modelled with multiplicative and monotonic functions, then a condition like the relative magnitude constraint may need to be met to make the addition valid.

Failing either condition implies that inferences derived from the combination of states are unsound.

7 Future Work

This paper has presented an alternative qualitative reasoning methodology, utilising qualitative histories to determine the effects of interactions within a system. The similarity between adding histories and path traversal through an envisionment was also identified. Several avenues for further development of this technique are apparent:

Diagnostic Monitoring: The techniques in this paper have been developed for, and incorporated into a program for the diagnostic monitoring of changes in patient state over time in the medical domain of acid-base disturbances [1], [2]. How they may optimally be used in conjunction with standard qualitative simulation remains to be determined. Other work with the monitoring of dynamic systems using qualitative simulation will serve as a useful benchmark e.g. [4].

Reasonableness conditions: A limitation of the current work is its application to non-linear models incorporating MULT constraints (and to a lesser extent monotonic constraints) where situations arise in which qualitative superposition cannot be performed. However, there is no reason why other instances where reasonable behaviour is exhibited cannot be found amongst these, and it is the identification of such instances that will enhance the applicability of the behaviour addition methodology.

Representing Structural Faults: For the techniques in this paper to work the underlying qualitative model that produced the interacting histories must be the same. Yet many abnormal system behaviours are generated by introducing structural changes into the model. One solution is to replace single faulted constraints with subsystems that reveal more detailed mechanism structures [12]. Thus, rather than altering the faulted constraint we can at the more detailed level hold parameters constant or perturb their values, thus reproducing the faulty behaviour at the higher level. This method may not always be possible and the practicality of performing such transformations needs to be more fully explored.

Comparative Analysis: Weld's work on comparative analysis of systems [15] takes a system behaviour and explains how that behaviour will alter after perturbation. The methodology is limited when the perturbation causes a topological change in the system behaviour. There is similarity between the determination of relative change in Weld's work, and determining the qualitative difference of two behaviours (Determining the difference of two behaviours is performed in an identical manner to adding two behaviours). Since there is no limitation on the topology of behaviours in the work presented in this paper, it is possible that it may have relevance to comparative analysis.

Appendix A

This appendix restates results demonstrating that for any linear system a set of linearly independent solutions can be found that allow the sum of any system solutions to also be a system solution, and that any nth order system of ODEs can be represented as a set of first order qualitative differential constraints.

A.1 Linearly Independent Solutions

A standard result from the study of ordinary differential equations is that for a given linear system of equations, the sum of any two solutions to the system is also a solution. This follows from the existence of linearly independent solutions. We can state this precisely in the following theorem [13]:

Theorem A.1: Let the functions $a_0, a_1, .., a_n$ be continuous on an interval I, with a_0 never zero. Let $u_0, u_1, .., u_n$ be n solutions of the nth order equation

$$a_0 y^n + a_1 y^{n-1} + ... + a_{n-1} y' + a_n y = 0$$

If the set $u_0, u_1, .., u_n$ is linearly independent, then every solution on I is of the form

$$y = c_1 u_1 + c_2 u_2 + .. + c_n u_n$$

where $c_1, .., c_n$ are constants. Thus, any linear combination of solutions is also a solution to the nth order equation. ❑

A.2 QSIM Systems are First Order

In the general case we have an nth order differential equation. Such an equation can be reduced to a series of first order equations, which each can be mapped onto a QSIM constraint. This is stated in the following theorem taken from [Kuipers 86]:

Theorem: A.2 Let

$$F[u(t), u'(t), .., u^n(t)] = 0$$

be an ordinary differential equation of order n, to be satisfied by a function $u:[a,b] \rightarrow \Re^*$, where F is defined only in terms of addition, multiplication, and negation, along with functions of continuous and strictly non-zero derivative. Then a set of parameters and constraints can be defined, corresponding with the equation, such that any reasonable function $u: \Re \rightarrow \Re$ which satisfies the equation also satisfies the set of constraints. ❑

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List of Figures

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3. Histories for an increment to both arms of the U-Tube

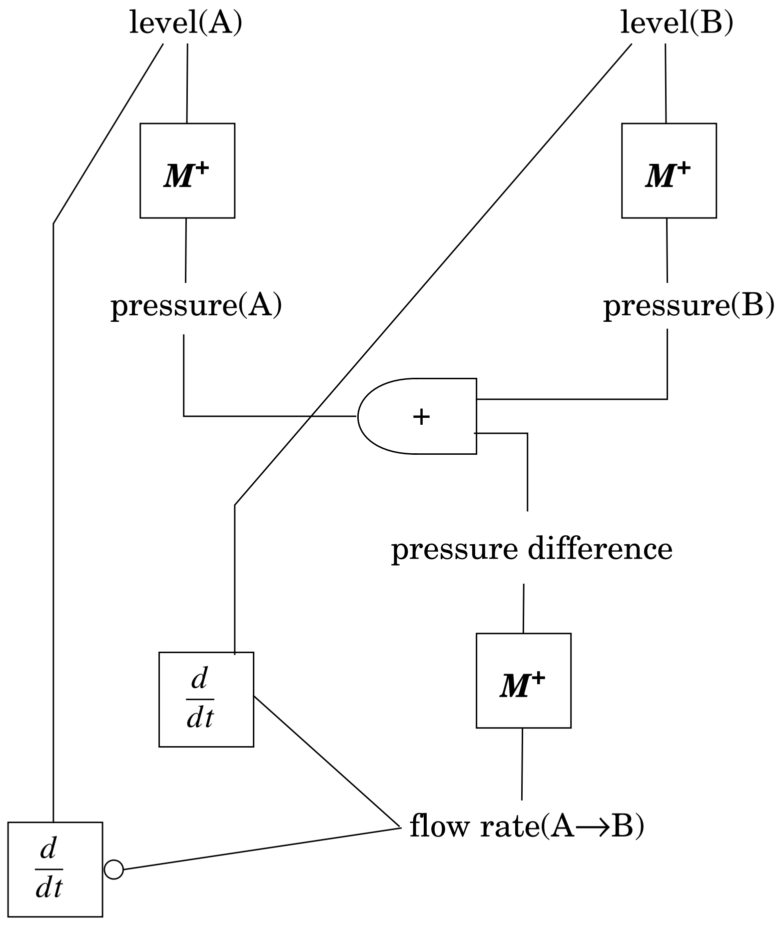

Figure 1. The QSIM U-Tube Model taken from Kuipers (1986).

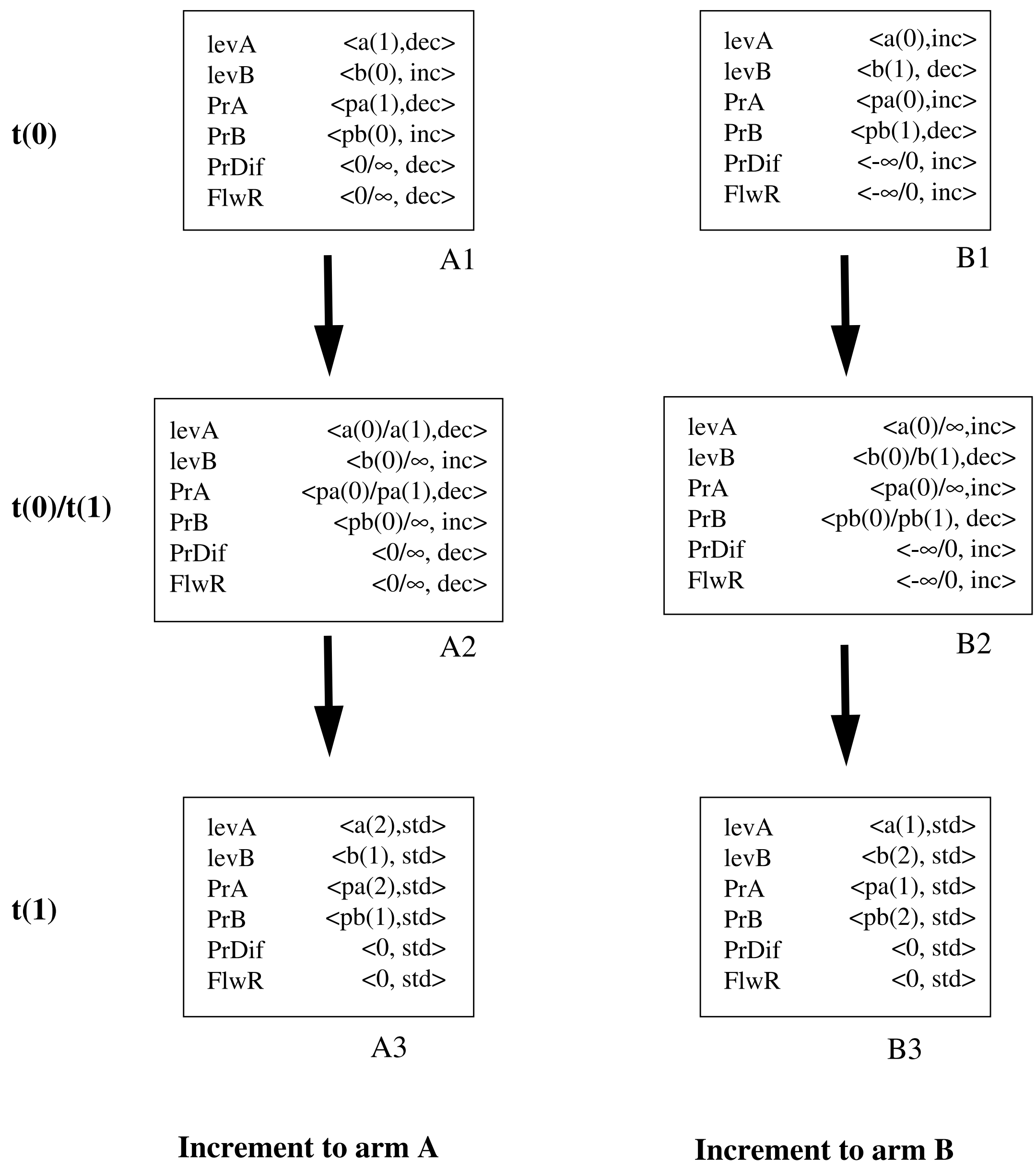

Figure 2.Histories for Fluid increments to arms A and B of the U-Tube

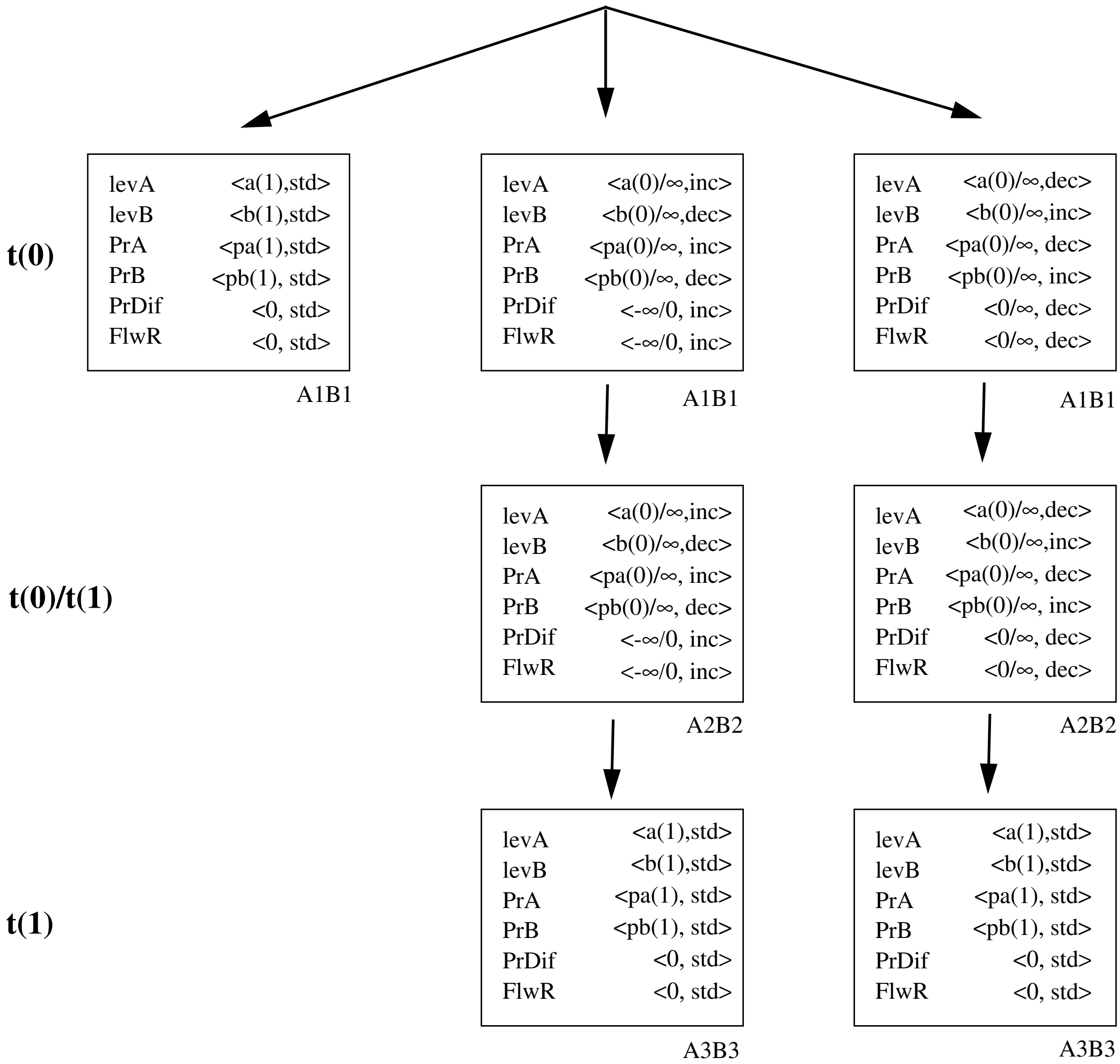

Figure 3. Histories for an increment to both arms of the U-Tube

Qualitative Superposition

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Abstract

Results are presented generalising superposition to non-linear systems by using qualitative differential equations. These are applied to allow the composition and decomposition of qualitative histories. Histories record the qualitative changes in a system over time, and they can be automatically generated by qualitative simulation. The qualitative superposition of such histories is shown to be identical to the qualitative simulation of interactions within linear systems, and many non-linear systems. The result of adding two histories is a partial envisionment for the system, and the recreation of the interaction history is a path traversal of the envisionment space. The technique is useful when a reasoning system needs to decompose an interaction history to identify the state of each contributing history, and when examples of a systems's behaviours exist as histories but no model is available. Formalising the limits to qualitative superposition also has implications for other forms of inference, such as the resolution of multiple causal inferences through a single parameter.

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1 Introduction

Histories record the qualitative changes in a system over time. They were first proposed by Hayes [9] as an improvement on the situational calculus for describing actions and change. A history was “a connected piece of space-time" which recorded events within a restricted spatial extent. The path a ball takes through the air or the response a physiological system makes to a disturbance can be recorded with histories.

Hayes noted that a history could be thought of as the extension or occurrence of a physical process. This relationship gained fuller expression in Forbus’ Qualitative Process Theory [6], in which physical systems were described in terms of the processes that occurred within them. QP Theory proposed a mechanism for the automatic generation of histories from such process descriptions. Since then, the generation of histories from system descriptions has been more formally identified as qualitative simulation, of which Kuipers’ QSIM [10] is a well known example. Histories in QSIM become solutions to models composed of qualitative differential constraints.

The relationship between histories and the underlying system model that produces them is reasonably well understood. Much less is known about how histories relate to one another, and how a reasoning system could use such relationships to make useful inferences. In Hayes’ Manifesto he noted that histories could relate in various ways. They could be adjacent, both spatially and temporally, or they could be hybrid in the sense that histories might interact. When two aircraft collide in mid air, we can think of each as having its own history, and at the moment of impact, a new composite history forms.

The *principle of superposition* is a well known mathematical property of linear systems [13]. It allows any two system states to be added to produce a new state description which is still consistent with that system. While representing a powerful formal technique, superposition is limited because it in general is inapplicable to non-linear systems. It will be show below that by relaxing system descriptions through the use of qualitative mathematical representations, superposition can in fact be applied to a wide range of non-linear systems. This result allows qualitative descriptions of system behaviours - histories - to be added qualitatively to reproduce behaviours consistent with the interaction of those histories.

Hayes [9] saw histories as a “basic ontological primitive” but work since then has focused largely on the mechanisms by which we can generate histories. Histories have thus become a by-product of inference procedures such as qualitative simulation. Qualitative superposition allows histories to once again be considered an important representational form in their own right.

1.1 Motivation

There are several situations in which the superposition of individual histories to determine interactions is advantageous.

Behavioural Decomposition: Qualitative simulation allows the overall behaviour of a system with interacting events to be determined. The simulation of interactions however loses information. We no longer know the individual state of each individual event within the interaction, just the effects of the union through the compound history. This becomes a problem when we want to know what individual processes are up to during the interaction. Such information is necessary for example in process monitoring applications. If the task is to monitor and treat a patient with multiple diseases, the state of each individual disease may need to be tracked within the overall observed behaviour and its individual response to therapy determined [2]. Such state information is inaccessible through normal qualitative simulation.

Behavioural Composition: If we can determine the behaviour of interactions within a system from the participating individual histories, then we only need to generate the individual histories to reason about interactions. Intuitively, this seems to be a simpler method of "naive" reasoning about interactions in physical systems than reasoning from first principles with a model. When using histories as the knowledge primitive, a reasoner can also use extra information to constrain the outcome of a situation. If, for example a bird collides with an aircraft, we can note that the mass of the aircraft completely dominates that of the bird. Of all the potential paths the two might take after the collision, we need only consider the history in which the aircraft's path is unchanged. A full qualitative simulation of the interaction system is avoided, all the inference occurring with the individual histories.

Reasoning without Models: One important consequence of using histories as a representational primitive is that in some circumstances one can make strong conclusions about systems that are as yet unmodelled. For example, empirical information about typical behaviours of a system may be available, but the system may be insufficiently characterised to have a useful qualitative model. Nevertheless qualitative superposition allows predictions to be made about the interactions of multiple behaviours from empirically identified single behaviours. One consequence of the lack of a qualitative model, as will be shown later, is that such predictions may be defeasible.

2 Reasoning with Histories - An Example

Consider the monitoring problem posed in the previous section. A physical system has a number of events occurring concurrently within it, and we wish to monitor the behaviour of each individual event along with the overall behaviour. The behaviour of fluid flowing within a U-Tube will serve as an example. A qualitative model of a U-Tube system is shown in Figure 1. The model is taken from [10] and is composed of a number of qualitative constraints between observable parameters (see section 3). The U-Tube is modelled as two fluid filled tanks (arm A and B) connected by a pipe. We model the fluid within the system with parameters for fluid levels, pressures and flows. These parameters are related to each other through qualitative mathematical constraints like addition, or statements of functional monotonicity.

Figure 1 near here

Qualitative simulations using QSIM on the U-Tube model following single increments of fluid being added to either arm A or B of the U-Tube are presented diagrammatically in Figure 2. Prior to the fluid increments, the system is quiescent, with equal positive fluid levels in each arm, recorded with the labels a(0) and b(0) respectively. To initialise the simulation for an increment to arm A we make the fluid level in arm A take a value greater than a(0) (labelled a(1) in Figure 2.). The simulation then discovers 3 distinct qualitative states, as a flow of fluid occurs between arm A and B, and a new quiescent state is reached with both fluid levels steady at new values a(2) and b(1), both higher than prior to the fluid increment. The label a(2) for the level in arm A lies somewhere between the level before and after initialisation. The simulation output for a single increment to arm B mirrors that for arm A.

Figure 2 near here

If QSIM is initialised to simulate the effects of the two fluid increments simultaneously, three separate histories are generated with quite different final state descriptions (see Figure 3). The histories correspond to the situations in which the amount of fluid added to arm A is equal to, greater than, and less than that added to arm B. The relative height of the initial fluid levels, and the direction of the net fluid flow between the arms is different in each of the three behaviours. In the first the history, the fluid increments are of equal size, so that no net flow occurs at this level of model resolution. In the middle most history the flow rate and pressure difference indicates that a net shift of fluid occurs from arm b to arm a indicating that the amount added to arm b was the greatest. The reverse scenario is represented in the third history.

Figure 3 near here

From the point of view of a reasoning program that wishes to track both contributing events, these three interaction histories are inadequate. Three problems are apparent:

- The loss of information about each individual event within the system description.
- The ambiguity of the possible outcomes. There is no clear way of informing QSIM which of the outcomes is preferred.
- Initialising a function prior to simulation is ambiguous if it is perturbed by *both* events and the initialisations conflict.

What is required is a way of mapping the states of the individual histories in Figure 2 to an interaction state in the histories in Figure 3. In fact such a mapping is possible. As foreshadowed in the introduction, it corresponds to the qualitative addition or superposition of the state descriptions taken from the individual histories to generate a state in the interaction history. Examining the rightmost interaction history in Figure 3, the state additions that produced it are:

$$a1b1 \rightarrow a2b2 \rightarrow a3b3$$

i.e. adding state a1 from the increment to arm A history to state b1 from the increment to arm B history can produce state a1b1 in the interaction history.

The three difficulties presented previously when simulating the progression of any two interacting histories have now been overcome. The first difficulty was the inability to track the effects of individual histories within the concurrent history. This information is now made explicit. The second problem, that of ambiguity of outcome, can be resolved by the use of information about the relative magnitude of the influence of each history on the interaction system during superposition. The process of state addition can be biased to favour solutions consistent with one history dominating the other. Previously there was no way of expressing such information in the standard QSIM formalism. Finally, the third problem of conflicting function initialisations can also be resolved through the use of relative magnitude information.

2.1 Overview

Given that there is some motivation for reasoning with individual histories as a complementary technique to full qualitative simulation, it remains to be determined under what conditions this might be valid. The QSIM algorithm and representation will be used as the exemplar system for the generation of histories throughout the rest this paper. There remains some controversy among researchers working in the area about whether a process ontology like QP theory, or a device ontology [3], [5] are appropriate representations for qualitative reasoning. QSIM makes no assumption about the origin of its qualitative constraint models, and so offers a low level description language for qualitative systems that could fit into either the process or device ontology [8]. The following sections will examine the superposition process in more detail. First, relevant portions of the QSIM formalism will be presented. Next the mathematical basis for qualitative superposition using the QSIM representation will be presented, including non-linear systems. An algorithm for generating composite histories after qualitative superposition is presented next. The paper concludes with a short discussion on reasoning with unmodelled behaviours, the role of superposition in causal reasoning, and future directions for this work.

3 Qualitative Simulation

Since QSIM is used as the reference simulation system, it will be necessary to briefly restate some definitions from Kuipers' original work [10]. Within the QSIM formalism, a system model consists of a number of parameters linked by constraint relationships. QSIM generates a state description for the system consistent with given initial conditions and operating ranges for these parameters, and generates all the valid state descriptions that might follow. A sequence of such states forms a history for the system being modelled. A summary of the QSIM definitions relevant to the results on superposition now follows.

Parameters: A physical system is characterised by a set of real valued parameters $f{:}[a,b] \rightarrow \Re^*$ which vary continuously over time, are continuously differentiable and have finitely many critical points in any bounded interval. Such parameters are called *reasonable functions*.

Landmarks: Every parameter is associated with a finite set of landmark values which must include 0, f(a), f(b), and the values of f(t) at each of its critical points. Landmark values are considered to be the only interesting values from which a qualitative state description need be drawn.

Time Points: Distinguished time points are those points where something important happens to a parameter, such as passing a landmark value or reaching a limit. Reasonable functions have a finite set of distinguished time points and landmarks values.

Qualitative State: The qualitative state of a parameter consists of its ordinal relation within the landmark values and its direction of change. If $l_1 < \bullet\bullet\bullet < l_k$ are the landmark values of $f{:}[a,b] \rightarrow \Re^*$, then for any $t \in [a,b]$, the qualitative state of f at t is $QS(f,t)$ and is a pair *(qval,qdir)* defined as follows:

$$qval = \begin{cases} l_j & if \quad f(t) = l_j \\ (l_j, l_{j+1}) & if \quad f(t) \in (l_j, l_{j+1}) \end{cases}$$

$$qdir = \begin{cases} inc & if \quad f(t) > 0 \\ std & if \quad f(t) = 0 \\ dec & if \quad f(t) < 0 \end{cases}$$

Qualitative Behaviours: The qualitative behaviour of f on $[a,b]$ is the sequence of qualitative states of f:

$$QS(f,t_0), QS(f,t_0,t_1), QS(f,t_1), \ldots, QS(f,t_{n-1},t_n), QS(f,t_n)$$

alternating between qualitative states at distinguished time points, and qualitative states on intervals between distinguished time points.

Systems: A system is a set of reasonable functions that are related by a set of qualitative differential equations, and the behaviour of the system is the union of the behaviours of its functions. Every qualitative state has a qualitative description, and that description can change at distinguished time points, and remains constant on the open intervals between them. The qualitative state of a system F of m functions is the m-tuple of individual states:

$$QS(F,t_i) = [QS(f_1,t_i), \ldots, QS(f_m,t_i)]$$

The qualitative behaviour of F is the sequence of states:

$$QS(F,t_0),QS(F,t_1),...,QS(F,t_n).$$

Constraints: A system model involves detailing functional relationships through a number of qualitative functional constraints. Constraints may be two or three placed predicates, and they restrict the qualitative values that may be assigned to functions. The QSIM constraints are:

- Add(f,g,h) iff $f(t) + g(t) = h(t)$
- Mult(f,g,h) iff $f(t) \times g(t) = h(t)$
- Minus(f,g) iff $f(t) = - g(t)$
- Deriv(f,g) iff $f'(t) = g(t)$
- $M^+(f,g)$ iff $f(t) = H(g(t));H(x)$ is strictly monotonically increasing
- $M^-(f,g)$ iff $f(t) = H(g(t));H(x)$ is strictly monotonically decreasing

We can consider these constraints as qualitative abstractions of Ordinary Differential Equations (ODEs). We can decompose any ODE into a set of simultaneous first order equations which can be replaced by a qualitative constraint. Thus a system model can be considered to be a set of q*ualitative differential equations* (QDEs). Any behaviour that satisfies an ODE must satisfy the corresponding QDE. Since many possible functions can map onto the same qualitative constraint however, a given QDE may be the abstraction of numerous ODEs.

4 Qualitative Superposition

Just as a qualitative model can be an abstraction of ordinary differential equations, a history can be regarded as an abstraction of the solution to such a set of equations. Two histories can add to reproduce an interaction history precisely when two qualitative solutions for a system can be added to produce another solution to a system[1]. It is a standard result from the study of ordinary differential equations that for a given linear system of differential equations, the sum of any two solutions to the system is also a solution. This superposition property follows from the existence of linearly independent solutions (Appendix A.1). Thus whenever two histories are combined and the histories are from a linear system, a true solution to that system results. However, this is simply not true in general for non-linear systems. Yet it is precisely nonlinear systems that are of interest since most of the physical systems that are of importance display non-linear behaviour.

One way of extending the property of superposability to non-linear systems is to weaken our requirements. In particular if we require superposition to only successfully operate on the qualitative behaviours of functions rather than their strict numeric values, we may still be able to derive useful inferences with it. The

1. In general, when two histories interact, this need not occur over all parameters in each history. This is especially the case when two subsystems communicate with each other through a small subset of their parameters. The interaction occurs only through the shared parameters, and this is specifically the area we are interested in. A simplifying assumption for the remainder of this discussion will be that non-shared parameters can be treated as a 'black box' when dealing with interacting subsystems and their interacting histories. We will only be interested in the inputs from the non-shared parameters into the shared system.

approach taken below will be to identify conditions in which qualitative solutions from non-linear systems behave as if the system they came from was linear.

Superposition has varying degrees of validity, depending on whether the system it is applied to is linear or non-linear, whether quantitative or qualitative solutions are desired, and on the nature of the non-linearity. Four validity classes can be identified:

Type 1. The Principle of Superposition applies uniformly for the quantitative values of all functions in a system. All linear systems fall into this class.

Type 2. Superposition applies for qualitative values of all functions in a system. Functional relations are preserved between the values produced by superposition.

Type 3. Functional relations cannot be maintained following qualitative superposition, but the resulting solution demonstrates the correct qualitative behaviour. This means that the sign of a function and its derivative can be predicted, but not its ordinal position in a landmark list.[1]

Type 4. Qualitative superposition is not supported.

4.1 General Approach

In the general case a physical system may be represented by an nth order differential equation. It was noted above that such an equation can be reduced to a series of first order equations, which can each be mapped onto QSIM constraints (Appendix A.2). The problem now reduces to identifying conditions under which we can add solutions for each of the qualitative constraints, and derive solutions that are still valid for these constraints. We shall call a constraint that satisfies such a condition a r*easonable constraint* and define it thus:

Definition 4.1: Let C be a qualitative constraint of the form $C(a,b)$ where a and b are reasonable functions. If $a_1, a_2,.. a_n$ are solutions for a and $b_1, b_2,.. b_n$ are solutions for b such that $C(a_1,b_1), C(a_2,b_2),.. C(a_n,b_n)$ are true, then if for any two constraint solutions x, y where $x \in [0,n]$ and $y \in [0,n]$, and $C(a_x + a_y, b_x + b_y)$ is true, then C is a *reasonable constraint*. ❑

A similar definition can be formulated for three placed predicates.

Definition: 4.2: If F is a system of reasonable constraints, then any behaviour produced with F is a r*easonable history*. ❑

4.2 Superposition with QSIM constraints

It can trivially be shown that the QSIM constraints MINUS, ADD, DERIV are reasonable since these constraints capture linear behaviour. Solutions will thus be of Type 1 (or 2 if so desired). Non-linear behaviours lie in the monotonic and multiplication constraints. We can demonstrate that the M^+ and M^- constraints support Type 2 qualitative superposition. This is not the case for the MULT constraint. However, making an assumption about the relative magnitude of the two solutions being combined is a sufficient condition for the MULT constraint to

1. In QSIM, corresponding values cannot be supported for Type 3 solutions.

allow Type 3 solutions to be produced. The following proofs demonstrate the degree of superposition possible for each of the MINUS, ADD, DERIV, M^+, M^- and MULT constraints. The constraint definitions given here are taken directly from [10].

4.2.1 The Linear Constraints

Definition 4.3: MINUS(f,g) is a two placed predicate on reasonable functions $f,g:[a,b] \rightarrow \Re^*$ which holds iff $f(t) = -g(t)$ for every $t \in [a,b]$

Proposition 4.3: If MINUS(f_1,g_1) and MINUS(f_2,g_2) are true then MINUS($f_1 + f_2, g1 + g_2$) holds.

Proof: Given $f_1 = - g_1$ and $f_2 = - g_2$, then

$$\begin{aligned} f_1 + f_2 &= -g_1 - g_2 \\ &= -(g_1 + g_2) \end{aligned}$$

❑

Definition 4.4: ADD(f,g,h) is a three placed predicate on reasonable functions $f,g,h:[a,b] \rightarrow \Re^*$ which holds iff $f(t) + g(t) = h(t)$ for every $t \in [a,b]$.

Proposition 4.4: If ADD(f_1,g_1,h_1) and ADD(f_2,g_2,h_2) then ADD($f_1 + f_2, g_1 + g_2,h_1 + h_2$) holds.

Proof: This result clearly follows from the definition of ADD. ❑

Definition 4.5: DERIV(f,g) is a two placed predicate on reasonable functions $f,g:[a,b] \rightarrow \Re^*$ which holds iff $f'(t) = g(t)$ for every $t \in [a,b]$

Proposition 4.5: If DERIV(f_1,g_1) and DERIV(f_2,g_2) are true then DERIV($f_1 + f_2,g_1 + g_2$) holds.

Proof: Given $g_1 = \frac{df_1}{dt}$ and $g_2 = \frac{df_2}{dt}$, then

$$\begin{aligned} \frac{d}{dt}(f_1 + f_2) &= \frac{df_1}{dt} + \frac{df_2}{dt} \\ &= (g_1 + g_2) \end{aligned}$$

❑

4.2.2 The Non-Linear Constraints

Definition 4.6: M^+ is a two placed predicate on reasonable functions $f,g:[a,b] \rightarrow \Re^*$. M^+ is true iff $f(t) = H(g(t))$ for all $t \in [a,b]$, where H is a function with domain $g([a,b])$ and range $f([a,b])$, differentiable and with $H'(x) > 0$ for all x in the interior of the domain.

The essential property captured in a monotonic increasing relationship between two functions is that they reach critical points at identical distinguished time points and that in the intervening regions they share the same direction of change [10] i.e. $\frac{f'}{g'} > 0$. It is this functional relationship that needs to be preserved after superposition.

Proposition 4.6: Given $M^+(f,g)$ and two solution pairs (f_1,g_1) and (f_2,g_2) then we can show $M^+(f_1 + f_2, g_1 + g_2)$ also holds. Thus we seek to demonstrate a functional relationship exists between (f_1,g_1) and (f_2,g_2) such that:

1. $\frac{f'_1 + f'_2}{g'_1 + g'_2} > 0$ between critical points (where $g'_1 + g'_2$ is defined) and
2 $f'_1 + f'_2$ and $g'_1 + g'_2$.reach critical points at the same time.

Proof:

1. Behaviour between critical points

Given that $f_1 = H(g_1)$ and $f_2 = H(g_2)$, we note that

$$f_1 + f_2 = H(g_1) + H(g_2)$$

$$f'_1 + f'_2 = H'(g_1) \times g'_1 + H'(g_2) \times g'_2$$

$$\frac{f'_1 + f'_2}{g'_1 + g'_2} = \frac{H'(g_1) \times g'_1}{g'_1 + g'_2} + \frac{H'(g_2) \times g'_2}{g'_1 + g'_2}$$

Let $A = \frac{H'(g_1) \times g'_1}{g'_1 + g'_2}$ and $B = \frac{H'(g_2) \times g'_2}{g'_1 + g'_2}$. We now evaluate the sign of the expression $\frac{f'_1 + f'_2}{g'_1 + g'_2}$ for values of g'_1 and g'_2:

Case 1.Let $g'_1 = 0$. Then $A = 0$ and $B = H'(g_2)$ which we know from Definition 4.6 is always positive.

Case 2.Let $g'_2 = 0$. Similarly, $B = 0$ and $A = H'(g_1)$ which is also always positive.

Case 3.Let $g'_1, g'_2 > 0$. Trivially both A and B are positive.

Case 4.Let $g'_1, g'_2 < 0$. Noting again that $H'(g_x) > 0$ both A and B evaluate to positive expressions.

Case 5.Let $g'_1 > 0, g'_2 < 0$ or $g'_1 < 0, g'_2 > 0$ and let x and y take the values 1 or 2. In both cases A + B is positive when

1. $|g'_x| > |g'_y|$ and
2. $|H'(g_x)| > |H'(g_y)|$. This second condition is equivalent to stipulating that $\left|\frac{f_x}{g'_x}\right| > \left|\frac{f_y}{g'_y}\right|$, or noting that condition 1 must also hold, simply that $|f_x| > |f_y|$.

These two conditions - $|g'_x| > |g'_y|$ and $|f_x| > |f_y|$ - together imply that the effect of one function's values dominates the others during the superposition and collectively form the *relative magnitude constraint*.

Cases 1 to 5 are summarised in Table 1. It may be possible to specify further conditions for superposition in Case 5 by closer examination of the function H. For example if we know it be linear, then the magnitude constraint may be dropped. If it is weakly linear such that $|g'_x| > |g'_y|$, but that $|H'(g_x)|$ is only minimally *less* than $|H'(g_y)|$ then the expression $H'(g_x) \times g'_x$ may still evaluate to be greater than $H'(g_y) \times g'_y$ and satisfy the sign of derivative requirement.

2. Behaviour at critical points.

We need to show that $f'_1 + f'_2$ and $g'_1 + g'_2$ reach zero at the same time. There are two cases at which $g'_1 + g'_2 = 0$ that need to be examined:

1. Both original functions reach critical points at the same time. By definition, when $g'_x = 0$ then $f'_x = 0$. Thus when f'_1 and f'_2 are zero, $g'_1 + g'_2 = 0$ and the expression $\frac{f'_1 + f'_2}{g'_1 + g'_2}$ is defined and equal to 0, implying that both $f'_1 + f'_2$ and $g'_1 + g'_2$ reach critical points when the original functions reach a critical point.
2 When $|g'_1| = |g'_2|$ but are opposite in sign, then $g'_1 + g'_2 = 0$. However $f'_1 + f'_2$ may be non-zero, making $\frac{f'_1 + f'_2}{g'_1 + g'_2}$ undefined. In this case one superposed function may reach a critical point without the other doing so. This case is avoided by the stipulation in Case 5 of the first part of this proof that a relative magnitude constraint must be enforcable for superposition to be possible when functions being added are of opposite sign.

A similar proof can be constructed for the M^- constraint. ❑

g_1' \ g_2'	+	0	-
+	✓	✓	*rmc*
0	✓	✓	✓
-	*rmc*	✓	✓

Table 1. Summary of Type 2 qualitative superposition results using the monotonic increasing constraint, given M^+(f1 + f2,g1 + g2). (rmc indicates that the relative magnitude constraint applies).

Definition 4.7: MULT(*f,g,h*) is a three placed predicate on reasonable functions $f,g,h:[a,b] \to \Re^*$ which holds iff $f(t) \times g(t) = h(t)$ for every $t \in [a,b]$.

In the general case if $f_i \times g_i = h_i$ and $f_j \times g_j = h_j$ then $(f_i + f_j) \times (g_i + g_j) \neq (h_i + h_j)$. However, we are interested in identifying situations in which the expression is still qualitatively true i.e. $h_i + h_j$ displays the same qualitative behaviour as $(f_i + f_j) \times (g_i + g_j)$ to produce a Type 3 prediction. This reduces to two subproblems:

- When is the sign of the function value preserved?
- When is the sign of derivative preserved?

Proposition 4.7: If MULT(f_1,g_1,h_1) and MULT(f_2,g_2,h_2) are true and we can specify the relative magnitude constraint, or a similar equivalence constraint i.e.:

1. $(|f_1(t)| > |f_2(t)|, |f'_1(t)| > |f'_2(t)|$ and $|g_1(t)| > |g_2(t)|, |g'_1(t)| > |g'_2(t)|)$
   or
   $(|f_1(t)| < |f_2(t)|, |f'_1(t)| < |f'_2(t)|$ and $|g_1(t)| < |g_2(t)|, |g'_1(t)| < |g'_2(t)|)$
2. $(|f_1(t)| = |f_2(t)|, |f'_1(t)| = |f'_2(t)|$ and $|g_1(t)| = |g_2(t)|, |g'_1(t)| = |g'_2(t)|)$

for all $t \in [a,b]$ then

- $sign(h_1 + h_2) = sign((f_1 + f_2) \times (g_1 + g_2))$ and

- $sign\left(\frac{d}{dt}(h_1 + h_2) \right) = sign\left(\frac{d}{dt}(f_1 + f_2) \cdot (g_1 + g_2) \right)$

Proof: Consider Case 1 (relative magnitude constraint). Note the result for two real numbers *a, b* that if $|a| > |b|$ then $sign(a + b) = sign(a)$.

$$\begin{aligned} & sign\,((f_1 + f_2) \cdot (g_1 + g_2)) \\ &= sign\,(f_1 \cdot g_1) \\ &= sign\,(f_1 g_1 + f_2 g_2) \\ &= sign\,(h_1 + h_2) \end{aligned}$$

Further, let

$$\begin{aligned} x &= (f_1 + f_2) \cdot (g_1 + g_2) \\ x' &= (f_1 + f_2)' \cdot (g_1 + g_2) + (f_1 + f_2) \cdot (g_1 + g_2)' \\ &= (f_1' + f_2') \cdot (g_1 + g_2) + (f_1 + f_2) \cdot (g_1' + g_2') \\ sign\,(x') &= sign\,(f_1' \cdot g_1 + f_1 \cdot g_1') \end{aligned}$$

Let

$$\begin{aligned} y &= h_1 + h_2 \\ &= (f_1 g_1 + f_2 g_2) \\ y' &= f_1' g_1 + f_1 g_1' + f_2' g_2 + f_2 g_2' \\ sign\,(y') &= sign\,(f_1' g_1 + f_1 g_1') \\ &= sign\,(x') \end{aligned}$$

A similar argument can be produced for the equivalence constraint defined in Proposition 4.7, case 2. ❑

5 Generating Composite Behaviours

Having demonstrated that qualitative superposition is valid for most systems expressible in the QSIM representation, it is now necessary to demonstrate how this property can be harnessed to generate interaction histories. Given two reasonable histories originating from the same system, how do we actually generate a composite behaviour to represent their interaction? There are two fundamental steps to this process:

1. *Superposition*: Adding qualitative states to produce all possible composite states,

2. *Assembly*: Concatenating composite states into a legal behaviour for the system.

The soundness of addition for reasonable histories means that any two such histories derived from the same set of QDEs can be used. An order preserving addition of states needs to be performed i.e. a state from one history is added to all the states in the other history with which it might possibly interact.

However, addition is an underconstrained process. Other solutions are also produced, along with the correct one, because of the inherent ambiguity of qualitative addition. Adding $QS_a(0/\infty,inc)$ to $QS_b(0/\infty,dec)$ produces either $QS_{a+b}(0/\infty,inc)$, $QS_{a+b}(0/\infty,std)$ or $QS_{a+b}(0/\infty,dec)$. Further, such solutions could be assembled into indefinitely long behaviours e.g.

$QS_{a+b}(<0/\infty,inc>,t_n)$, $QS_{a+b}(<0/\infty,std>,t_n/t_{n+1})$, $QS_{a+b}(<0/\infty,dec>,t_{n+1})$, $QS_{a+b}(<0/\infty,std>,t_{n+1}/t_{n+2})$, $QS_{a+b}(<0/\infty,inc>,t_{n+2})$, $QS_{a+b}(<0/\infty,std>,t_{n+2}/t_{n+3})$,…

This problem is called "chatter" [11], and results from the inherent ambiguity of qualitative simulation, as well as problems with the locality of transition value selection in QSIM. These problems can be controlled by:

- Filtering composite states that are not valid system behaviours during the qualitative addition.
- Enforcing behavioural continuity when assembling sum states into new histories.

Both of these techniques will now be explored in more detail.

5.1 Superposition - Creating and Filtering Composite States

Although adding two qualitative states taken from two histories will produce unwanted states along with the real ones, there are at least two ways that such states can be eliminated:

1. *Using the original system constraints*. The sum states can be filtered through the system model that produced the original histories i.e. a composite behaviour must be legal for the system that produces it.
2. *Making an assumption about relative magnitude*s. Knowing the magnitude of the effect each history has on the interaction allows a decision about which history dominates during the addition to be made. While this is required to establish reasonableness for some non-linear systems, it also eliminates spurious solutions during state addition.

Rather than generating all possible states derivable from qualitative addition and then filtering them, it is easier to filter them during the process of qualitative addition. In effect, a single pass qualitative simulation is performed using the system model along with addition constraints between the behaviours of interest.

Further, rather than performing a full permutation of state additions, only those states that preserve functional or temporal continuity need to be added (see Section 5.2). In particular, transients from contributing histories cannot exist more than momentarily. Such transients, when they occur, are present as initial states of the histories being added. We avoid adding initial states with transients to states from the other history other than its initial state. Hence states a1 and b1 in Figure 2. can only be assembled into composite state a1b1. A state a1b2 would imply that a history of a1b1 → a1b2 was possible, implying incorrectly that the a1 transient could extended over the two qualitative states, a point and an interval.

The state generation algorithm first selects the states to be added, and then performs the addition by simulation. Once two states have been selected for addition, the superposition by qualitative simulation occurs in three stages. Assume a system of functions F, a set of constraints C, and two reasonable histories H_1 and H_2 from F and C. For each pair of qualitative states to be added we perform the following steps:

Step 1 - Composite Model Assembly:

1. For each function $f \in F$ create three copies labelled f_1, f_2, f_{sum}. Call the system of new functions F_{sum}.
2. For each function triplet so created, generate an *ADD* constraint $ADD(f_1,f_2,f_{sum})$.
3. For each constraint $c \in C$ generate a copy *csum* between the sum functions F_{sum}. Call the new *ADD* constraints and the copy of C together C_{sum}.
4. Landmark values, domain restrictions and corresponding values for each $f \in F$ are transferred for each copy function $f_{sum} \in F_{sum}$.

Step 2 - Composite Model Initialisation:

For each function $f_1 \in F_{sum}$ initialise the function value to its value in H_1. Repeat the procedure for each $f_2 \in F_{sum}$ using values in H_2.

Step 3 - Qualitative Simulation on the Composite Model:

Using QSIM, perform a single iteration of the algorithm on the composite model F_{sum} and C_{sum} to find legal values for each *fsum* and assemble these values into legal state descriptions for F_{sum}. Each coherent set of value assignments to the members of F_{sum} is a solution to the addition.

During the qualitative simulation on the composite system model, a restricted table of possible solutions for the *ADD* constraint can be used when one state in the addition is known apriori to dominate the addition (Tables 2 and 3). Such knowledge is needed when superposition requires a relative magnitude constraint, but can also be used to assist in reducing the possible states derived during the simulation.

The result of this process for two states taken from two reasonable histories is all the possible interaction states derivable for those two states. The next step in the

+ *P* / *Q*	>0	0	<0
>0	>0	>0	>0
0	-	0	-
<0	<0	<0	<0

Table 2. Landmark addition table with relative magnitude assumption |*Q*| > |*P*|

+ *deriv(P)* / *deriv(Q)*	inc	std	dec
inc	inc	inc	inc
std	-	std	-
dec	dec	dec	dec

Table 3. Direction of change addition table with relative magnitude assumption |*deriv(Q)*| >|*deriv(P)*|

superposition process is the linking of these states into complete histories representing interaction behaviours.

5.2 Path Assembly - Enforcing Behavioural Continuity

The technique of e*nvisioning* generates all the possible state descriptions for a system model, based on a set of background assumptions. Any specific history for the system corresponds informally to a path through that envisionment. If we have two reasonable histories, and do a permuted state addition between the two, a state space is defined which is equivalent to or is some subset of a full envisionment. The task of assembling an interaction history from this new state space thus becomes one of path traversal.

A *Logic of Occurrence* was described by Forbus [7], which presented a formalised relationship between histories and envisionments. In particular, the task of selecting a path through the envisionment for a partially created history was described. In our case, if we have selected a state from the partial envisionment created by history addition, we need to determine which states are possible next states for the interaction history. Often additional information about the speed of completion of each history and the synchronicity of transition to subsequent states will be available. Such information greatly constrains the search for a path through the partial envisionment.

A method for path traversal that allows information about the individual interacting behaviours to be utilised will be presented below. In accordance with Forbus' Logic of Occurrence, we assume here that the histories we will generate are finite - that they will terminate at some time in the future. We will also assume Forbus' terminology, which is briefly defined now.

Envisionment: An envisionment ε represents all possible qualitative states a particular system may take on and all legal transitions between them.

Transition functions: These describe all transitions involving a state s. State $s_i \in$ *Befores(s)* when the envisionment contains a transition from s_i to s. State $s_i \in$ *Afters(s)* when the envisionment contains a transition from s to s_i.

Paths: A Path is a sequence of states $s_i,..,s_n$ such that $s_{i+1} \in Afters(s_i)$.

Status of States: Consider a history H involving envisionment ε. For every state s ∈ States(ε), exactly one of the following is true: *possible(s,H), required(s,H), excluded(s,H)*. Simply, a state is *possible* if it may appear in a history, *excluded* if it never occurs, and *required* if it must appear.

Forbus introduces mechanisms for handling cycles, and avoiding impossible cyclic behaviours. These techniques for path traversal of envisionments are directly applicable to the state space generated by summing behaviours, and will not be reproduced here.

5.2.1 Path creation

Unlike the envisionments in Forbus' work, states produced by superposition are not explicitly linked. The process of history generation by path traversal must thus include a step that identifies candidate next states. Given a current state *current(s)* in a history H the next states in the path is defined as:

$$\forall\ current(s)\ \ \exists\ next(r) \rightarrow r \in States(\varepsilon), \neg\ excluded(r,H).$$

We need to identify next states that are legal transitions, before we can decide whether they are excluded. Possible states are identified using restrictions imposed by temporal continuity, and excluded states by application of rules that ensure functional continuity through the assembled history.

5.2.2 Temporal Continuity

Next states can be identified with knowledge about the temporal behaviour of the original contributing histories. In particular, each sum state carries with it the labels of the distinguished time points or intervals taken from its contributing states. Next states are identified with reference to that label.

The search for legal next states can be coded in several rules. Assume two reasonable histories H_i and H_j and states drawn from each history s_i and s_j. Let s_i have qualitative description $QS(f_i,t_i)$ and s_j have $QS(f_j,t_j)$, and call the resulting set of summation states $S_{i,j}$. Each state $s \in S_{i,j}$ is then labelled $QS(f_{i+j},t_i,t_j)$.

Rule 1 - Temporal Progression: For a state $s \in S_{i,j}$ which is the current last state in a history H, and has qualitative state $QS(f_{i+j},t_i,t_j)$, then

$$possible(H,s) = \{QS(f_{i+j},t_i,t_j), QS(f_{i+j},t_{i+1},t_j), QS(f_{i+j},t_i,t_{j+1}), QS(f_{i+j},t_{i+1},t_{j+1})\}$$

The Temporal Progression rule prevents local cycles being generated, and enforces the intuition that for the interaction history to progress to a new qualitative state, then either or both of the contributing histories must also progress. Note that two sequential states in a path may be the result of the same state addition i.e. a state with a $QS(f_{i+j},t_i,t_j)$ label can be a next state for a state with the same label. This reflects the possibility that the addition of two states can produce more than one interaction state.

The Temporal Progression rule can be specialised in some cases to utilise additional knowledge about the relationship between interacting histories. In particular information about synchronicity of history evolution and speed of completion may be utilised to constrain the process of path creation:

Rule 2 - Synchronous Progression: Assume two reasonable histories H_i and H_j have equal path length. For a state $s \in S_{i,j}$ which is the current last state in a history H, and has qualitative state $QS(f_{i+j},t_i,t_j)$, then

$$possible(H,s) = QS(f_{i+j},t_{i+1},t_{j+1})$$

If we know that the interacting histories will both progress to a new distinguished time point in a synchronous manner, then we can exclude asynchronous combinations.

Rule 3 - Asynchronous Progression: For a state $s \in S_{i,j}$ which is the current last state in a history H, and has qualitative state $QS(f_{i+j},t_i,t_j)$, then

$$possible(H,s) = \{QS(f_{i+j},t_{i+1},t_j), QS(f_{i+j},t_i,t_{j+1})\}$$

The Asynchronous Progression Rule enforces asynchronous progression of the contributing histories.

Rule 4 - Speed of Progression: Assume two reasonable histories H_i and H_j of equal path length where for any interval (t_i,t_{i+1}),

$$duration(H_i,t_i,t_{i+1}) < duration(H_j,t_i,t_{i+1}).$$

Then, for any distinguished time point with state $s \in S_{i,j}$ and with qualitative value

$$QS(f_{i+j},t_a,t_b),\ t_a \geq t_b.$$

The speed rule ensures that the faster of the two contributing histories reaches its final state first. In applying the Speed rule, we are making assumptions about which particular path among several alternatives we take to create our composite history. The effect of this rule is not to eliminate possible histories, as the synchronicity rules do, but to eliminate alternative paths for each history. It reduces the size of the search space in the sum envisionment, not the possible histories the envisionment contains.

5.2.3 Functional Continuity

Once the set of possible next states has been identified, the path creation algorithm chooses only those states that maintain functional continuity. For example, two states would not be temporally adjacent in a history if a functional discontinuity exists between them e.g. QS(<0,inc>,t_i), QS(<0,dec>,t_{i+1}) would not be allowed.

Further, continuity determines whether adjacent states can have the same description. In particular, a transition from an interval to a time point should not have the same qualitative state since distinguished time points represent critical points, but the reverse is allowable [10].

Functional continuity applies not just to the history being assembled, but to the histories that contribute to the assembly. The process of selecting possible next states ensures that the contributing histories are traversed in a legal temporal sequence. Along with the composite history, we need to ensure that states from the contributing histories are not prolonged unnecessarily. As indicated in Section 5.1 transients in the histories contributing to the superposition can only exist for an instant and state combinations that would prolong them are filtered during the initial state generation.

Functional continuity could also be invoked in the process of *value recovery*. As we saw in section 4.2.2, there are some values for functions in a monotonic relationship which prevent superposition. However, by looking ahead to the next state, it may be that superposition is once again possible. Where this is the case, the non-superposable state might be recreated by interpolation between its predecessor and successor states, relying on the fact that function values must be continuous.

6 Qualitative Superposition - some consequences

A number of consequences arise from the results in this paper which both expand the repertoire of qualitative reasoning systems and also impose limits on some types of causal reasoning. Two particular issues are explored here - the notion that histories can be used independently of deeper qualitative models, and the validity of the assumption that causal influences can be considered linearly independent.

6.1 Histories without Models

In some circumstances we might want to make predictions based on histories that were not generated from a model. This might be the case when empirical information about a system's behaviour is available but there is insufficient understanding of its mechanisms to construct a qualitative model.

In such circumstances we do not know whether the histories we are manipulating arise from linear or non-linear systems, or in fact if indeed the histories come from the same system. If the behaviours represent faulty behaviours, it may be that a different fault model produced each behaviour. Different models clearly imply different sets of system equations, and in such situations summing solutions is meaningless. A consequence of these uncertainties is that, although a mechanism now exists to produce predictions about interactions directly from behaviours, such predictions must be considered *defeasible*. Recall that although QSIM generates some behaviours that are not true for a system, it will always generate all possible true behaviours. We now have no such guarantee.

For a non-modelled behaviour to be useful, it should be *well-formed*. By this we mean that it displays all the features of a true model generated behaviour such as continuity of function behaviour. Further, it is ideal if a relative magnitude assumption between the summed behaviours can be made. This is so for two reasons. The absence of a model means that the addition of qualitative states is underconstrained - we do not have a model to filter out ill-behaved solutions. The use of relative magnitude information sufficiently constrains the addition process in such circumstances. Secondly, we should make a worst case assumption that the underlying system model is non-linear. By assuming that one history dominates the superposition we ensure that valid predictions are made for systems that contain multiplicative constraints between functions.

6.2 Combining Causal States

Causality may be represented as the temporal progression of states within a system. For example, take a CASNET [14] link between two states involved in the progression of the eye disease glaucoma:

Elevated Pressure Transmitted to Optic Nerve Head $\rightarrow$ (0.9)

Decreased Blood Supply to Optic Nerve Head

Paths through causal networks composed of such states are direct analogues of the qualitative histories discussed in this paper. It may be the case that the effects of two states exist concurrently, and are resolved to produce a third in a network - a process analogous to superposition. If we do view the combination of states as qualitative superposition, then the results presented earlier establish that a number of conditions need to be met prior to combination. In particular the states must be:

1. *Derived from the same structural model*: It is meaningless to use superposition on states that come from different models of a system - the procedure assumes that they represent solutions to the same set of qualitative differential equations. Consequently when states are combined they must belong to the same model. States that come for example, from different fault models of a mechanism cannot be combined.
2. *Satisfy conditions of reasonableness*: If the system that produced the causal states is modelled with multiplicative and monotonic functions, then a condition like the relative magnitude constraint may need to be met to make the addition valid.

Failing either condition implies that inferences derived from the combination of states are unsound.

7 Future Work

This paper has presented an alternative qualitative reasoning methodology, utilising qualitative histories to determine the effects of interactions within a system. The similarity between adding histories and path traversal through an envisionment was also identified. Several avenues for further development of this technique are apparent:

Diagnostic Monitoring: The techniques in this paper have been developed for, and incorporated into a program for the diagnostic monitoring of changes in patient state over time in the medical domain of acid-base disturbances [1], [2]. How they may optimally be used in conjunction with standard qualitative simulation remains to be determined. Other work with the monitoring of dynamic systems using qualitative simulation will serve as a useful benchmark e.g. [4].

Reasonableness conditions: A limitation of the current work is its application to non-linear models incorporating MULT constraints (and to a lesser extent monotonic constraints) where situations arise in which qualitative superposition cannot be performed. However, there is no reason why other instances where reasonable behaviour is exhibited cannot be found amongst these, and it is the identification of such instances that will enhance the applicability of the behaviour addition methodology.

Representing Structural Faults: For the techniques in this paper to work the underlying qualitative model that produced the interacting histories must be the same. Yet many abnormal system behaviours are generated by introducing structural changes into the model. One solution is to replace single faulted constraints with subsystems that reveal more detailed mechanism structures [12]. Thus, rather than altering the faulted constraint we can at the more detailed level hold parameters constant or perturb their values, thus reproducing the faulty behaviour at the higher level. This method may not always be possible and the practicality of performing such transformations needs to be more fully explored.

Comparative Analysis: Weld's work on comparative analysis of systems [15] takes a system behaviour and explains how that behaviour will alter after perturbation. The methodology is limited when the perturbation causes a topological change in the system behaviour. There is similarity between the determination of relative change in Weld's work, and determining the qualitative difference of two behaviours (Determining the difference of two behaviours is performed in an identical manner to adding two behaviours). Since there is no limitation on the topology of behaviours in the work presented in this paper, it is possible that it may have relevance to comparative analysis.

Appendix A

This appendix restates results demonstrating that for any linear system a set of linearly independent solutions can be found that allow the sum of any system solutions to also be a system solution, and that any nth order system of ODEs can be represented as a set of first order qualitative differential constraints.

A.1 Linearly Independent Solutions

A standard result from the study of ordinary differential equations is that for a given linear system of equations, the sum of any two solutions to the system is also a solution. This follows from the existence of linearly independent solutions. We can state this precisely in the following theorem [13]:

Theorem A.1: Let the functions $a_0, a_1, .., a_n$ be continuous on an interval I, with a_0 never zero. Let $u_0, u_1, .., u_n$ be n solutions of the nth order equation

$$a_0 y^n + a_1 y^{n-1} + ... + a_{n-1} y' + a_n y = 0$$

If the set $u_0, u_1, .., u_n$ is linearly independent, then every solution on I is of the form

$$y = c_1 u_1 + c_2 u_2 + .. + c_n u_n$$

where $c_1, .., c_n$ are constants. Thus, any linear combination of solutions is also a solution to the nth order equation. ❑

A.2 QSIM Systems are First Order

In the general case we have an nth order differential equation. Such an equation can be reduced to a series of first order equations, which each can be mapped onto a QSIM constraint. This is stated in the following theorem taken from [Kuipers 86]:

Theorem: A.2 Let

$$F[u(t), u'(t), .., u^n(t)] = 0$$

be an ordinary differential equation of order n, to be satisfied by a function $u:[a,b] \rightarrow \Re^*$, where F is defined only in terms of addition, multiplication, and negation, along with functions of continuous and strictly non-zero derivative. Then a set of parameters and constraints can be defined, corresponding with the equation, such that any reasonable function $u: \Re \rightarrow \Re$ which satisfies the equation also satisfies the set of constraints. ❑

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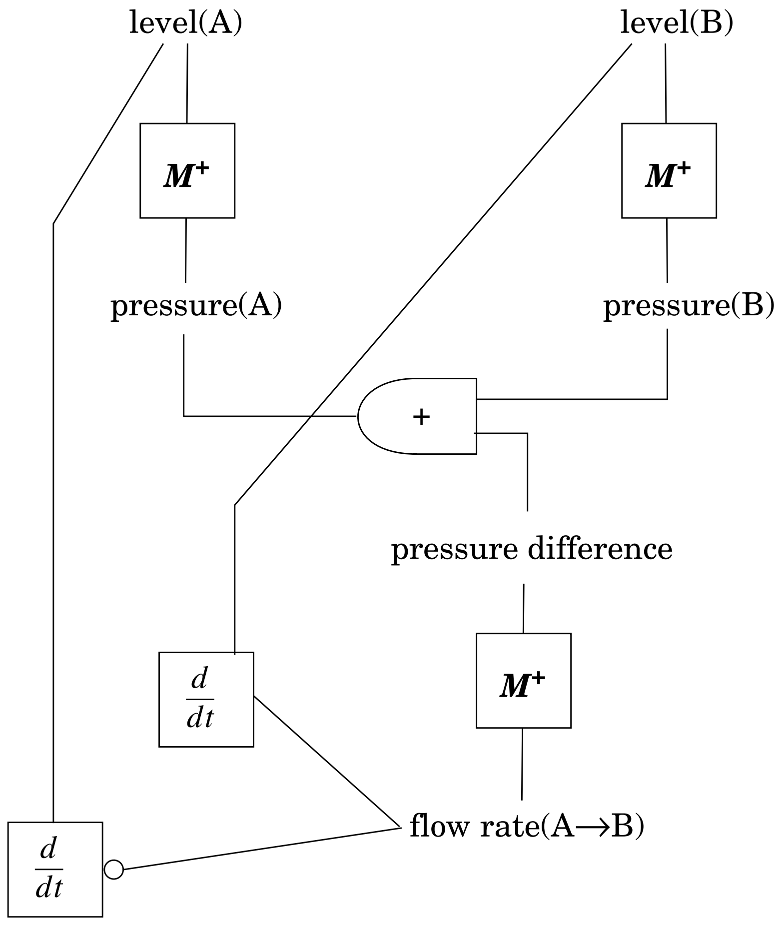


Figure 1. The QSIM U-Tube Model taken from Kuipers (1986).

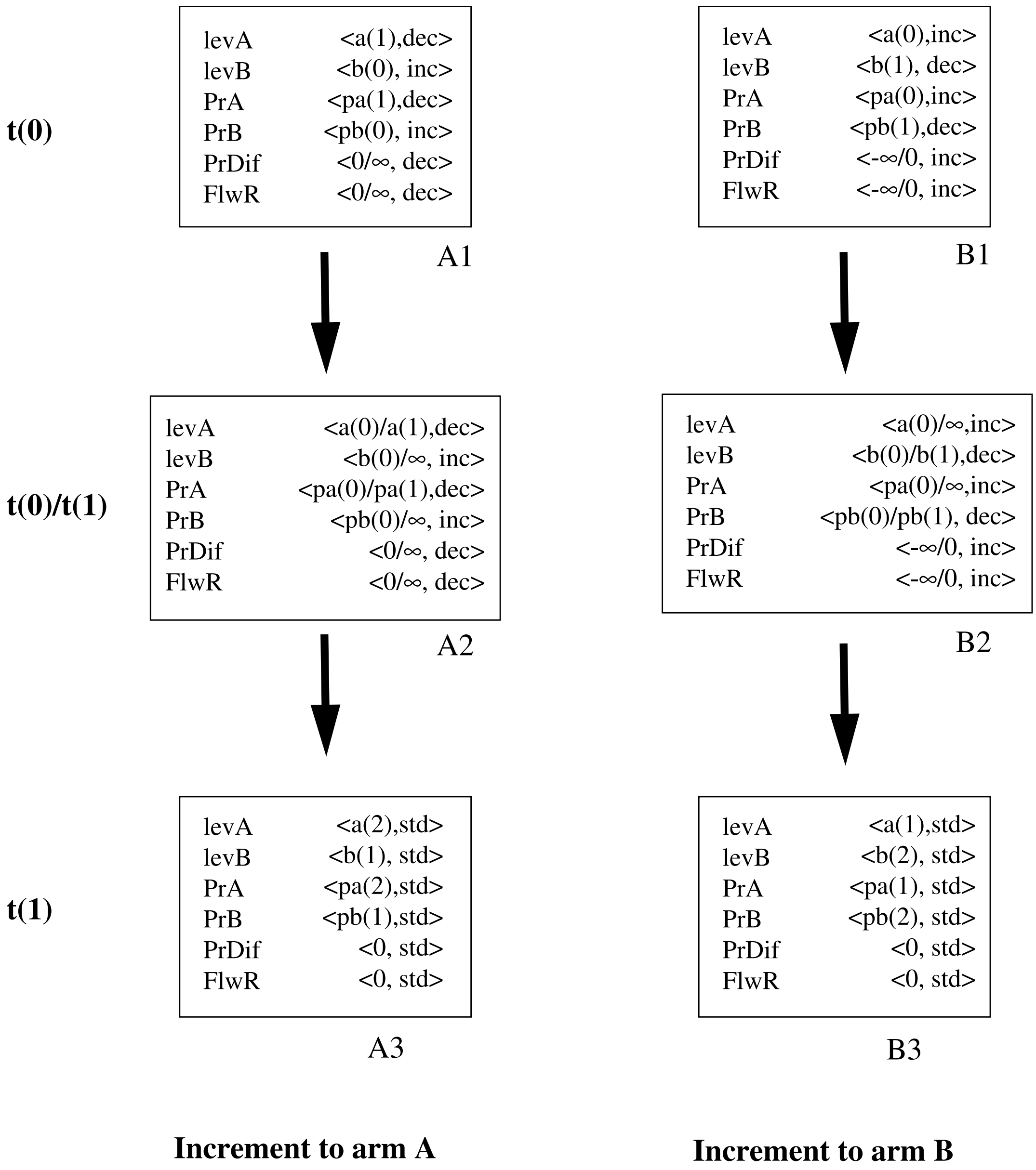
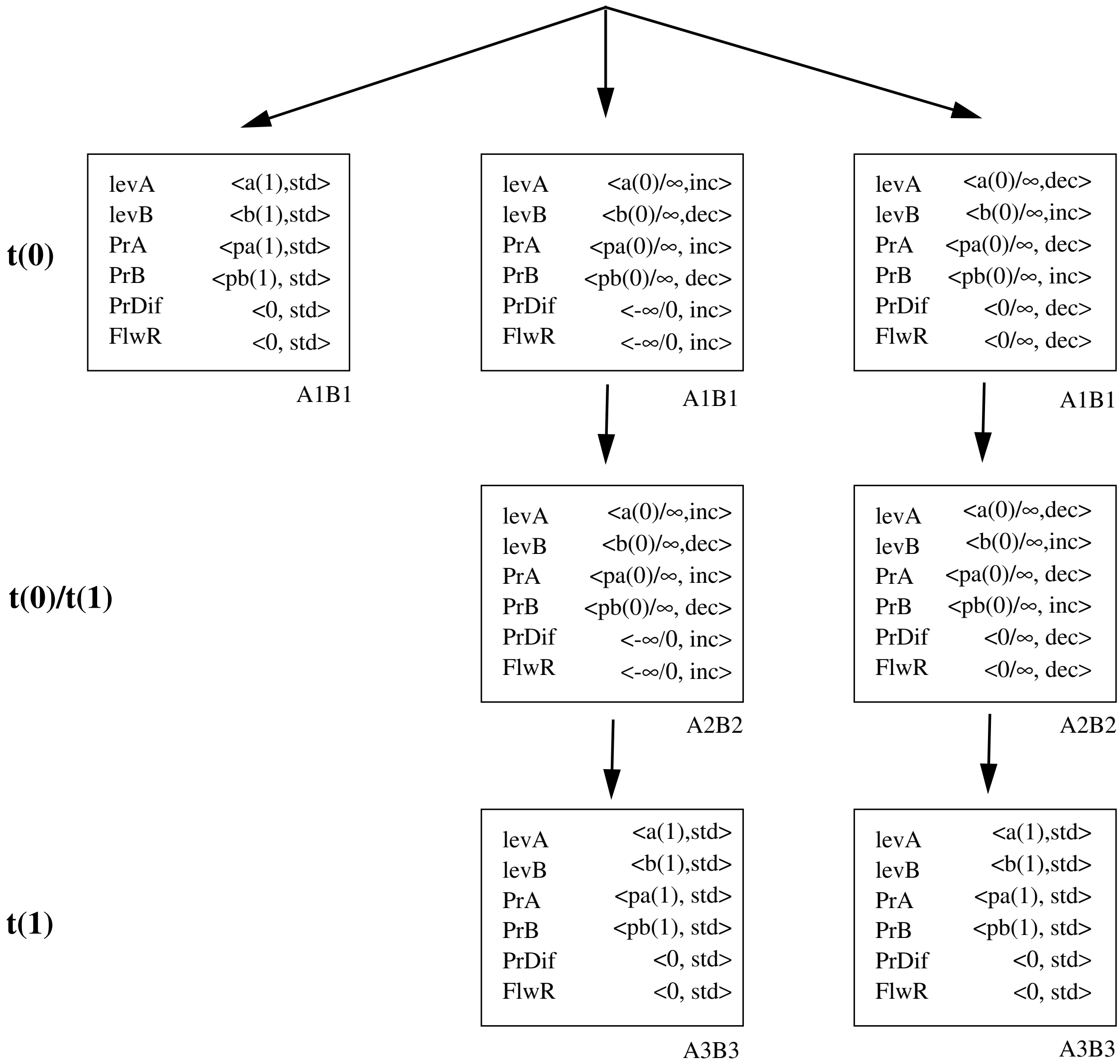


Figure 2.Histories for Fluid increments to arms A and B of the U-Tube



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